An Automata Theoretic Approach to Rational Tree Relations

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Motivation: Rational Relations over Words

There exist several equivalent descriptions of rational word relations, e. g.

- rational expressions:
  - for instance $R = \{(v, uv) \mid u, v \in \Sigma^*\}$ is definable by $\left(\binom{\varepsilon}{a} + \binom{\varepsilon}{b}\right)^* \cdot \left(\binom{a}{a} + \binom{b}{b}\right)^*$
- asynchronous automata (also called multitape automata):
  - (these have transitions of the form $p \xrightarrow{a_1/\ldots/a_n} q$ in $Q \times (\Sigma_1 \cup \{\varepsilon\}) \times \ldots \times (\Sigma_n \cup \{\varepsilon\}) \times Q$)

Rational Tree Relations

- Goal: Generalization to trees with the alternative descriptions by rational expressions and asynchronous automata.
- Starting point: Definition via rational expressions over trees (J.-C. Raoult (1997)).
- An automata theoretic approach is still missing.
  We define asynchronous tree automata recognizing exactly Raoult's relations.

Rational Tree Expressions

Jean-Claude Raoult.
Rational tree relations.
Bull. Belg. Math. Soc., 4(1): 149–176, 1997.

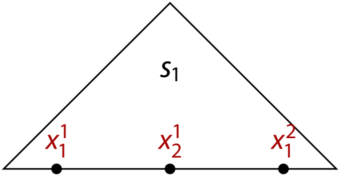
Definition

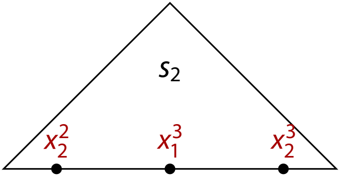
The classes Rat_n of rational tree relations are defined inductively:

- Each finite n-ary tree relation is in Rat_n.
- $R \in \mathrm{Rat}_n \wedge S \in \mathrm{Rat}_n \implies R \cup S \in \mathrm{Rat}_n$.
- $R \in \mathrm{Rat}_n \wedge |X| = m \wedge S \in \mathrm{Rat}_m \implies R \cdot_X S \in \mathrm{Rat}_n$.
- $R \in \mathrm{Rat}_n \wedge |X| = n \implies R^{*_X} \in \mathrm{Rat}_n$.

Concatenation via a Multivariable X

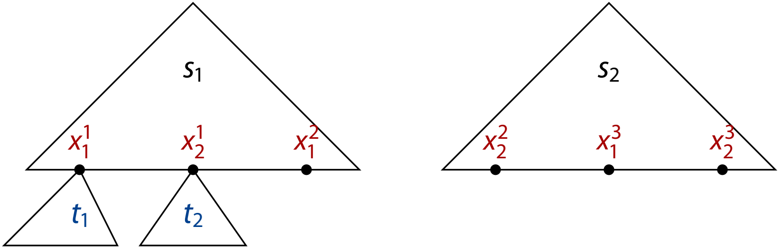
$\{(s_1, s_2)\} \cdot_X \{(t_1, t_2), (t'_1, t'_2)\}$ with respect to a multivariable $X = x_1 x_2$:





Concatenation via a Multivariable X

$\{(s_1, s_2)\} \cdot_X \{(t_1, t_2), (t_1', t_2')\}$ with respect to a multivariable $X = x_1 x_2$:



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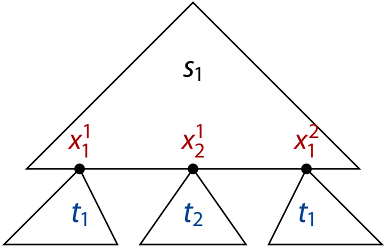

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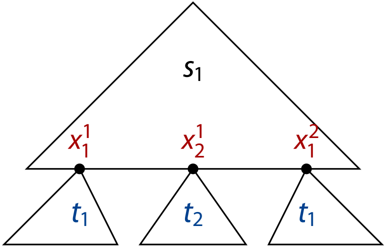
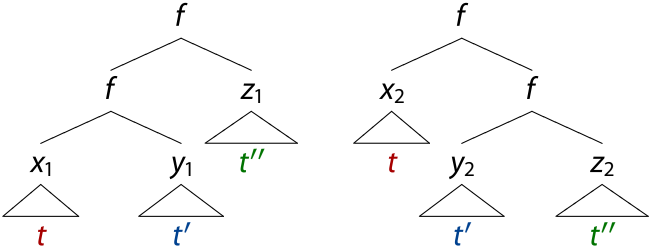


($2^3 = 8$ possibilities)

Example 1

Rational expression

$(ffx_1y_1z_1, fx_2fy_2z_2) \cdot_{x_1x_2} (t, t) \cdot_{y_1y_2} (t', t') \cdot_{z_1z_2} (t'', t'')$

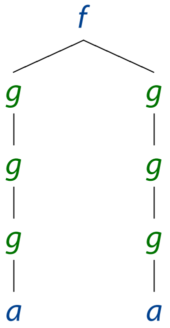


Right rotation is definable by a rational expression.

Example 2

Rational expression

$(f x_1 x_2) \cdot_{x_1 x_2} (g x_1, g x_2)^{*_{x_1 x_2}} \cdot_{x_1 x_2} (a, a)$

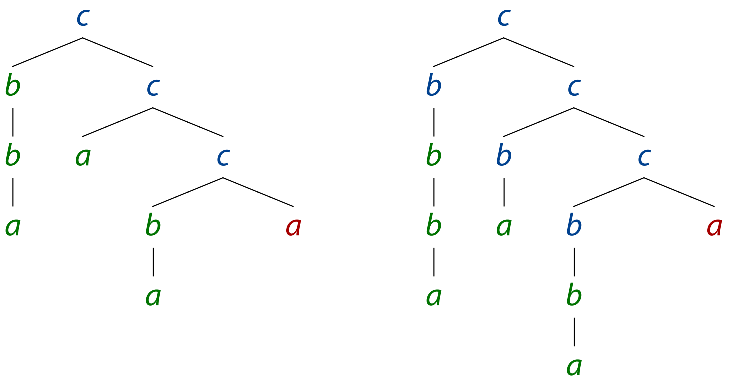


Consequence: Unary rational tree relations do not coincide with regular tree languages.

Example 3

Rational expression

$(cx_1y_1, cbx_2y_2)^{*_{y_1y_2}} \cdot_{y_1y_2} (a, a) \cdot_{x_1x_2} (bx_1, bx_2)^{*_{x_1x_2}} \cdot_{x_1x_2} (a, a)$



An unbounded number of multivariable instances can be generated.

Asynchronous Tree Automata

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- Mechanism analogous to multivariables.
  - Macro states: e. g. $\mathfrak{p} = (p_1, p_2, p_3)$, $\mathfrak{q} = (q_1, q_2)$
  - Finite set of macro states: e. g. $\mathfrak{Q} = \{\mathfrak{p}, \mathfrak{q}\}$

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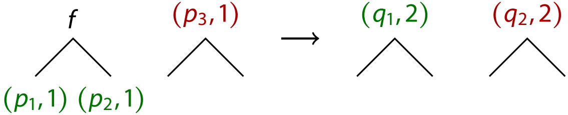
- Mechanism analogous to multivariables.
  - Macro states: e. g. $\mathfrak{p} = (p_1, p_2, p_3)$, $\mathfrak{q} = (q_1, q_2)$
  - Finite set of macro states: e. g. $\mathfrak{Q} = \{\mathfrak{p}, \mathfrak{q}\}$
- Instances of macro states have to be distinguishable.
  - Combining states with formal indices in the transitions:
    $((p_1, i), (p_2, i), f, (q_1, j))$, $((p_3, i), \varepsilon, (q_2, j))$
  - Instantiation in the bottom-up run with natural numbers:



  - Only complete instances of macro states are left and reached.

Decidability Results

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- Undecidability Results are inherited from rational word relations.
  - $R \cap S = \emptyset$
  - $R \subseteq S$
  - $R = S$

  It is undecidable whether a rational tree language is regular.

Drawbacks

- Restricted to unary sets, rational tree relations do not coincide with the class of regular tree languages.

- Non-closure under composition (for binary relations).

- When a binary relation $R \subseteq A \times B$ is considered as transduction $\tau_R : A \to \mathcal{P}(B)$, rational tree relations do not preserve regular tree languages.

  (e. g. the image over $T_\Sigma \times \{f(g^n a, g^n a) \mid n \in \mathbb{N}\}$ is not regular.)

Restrictions of Rational Tree Relations

Restrictions of rational tree relations to overcome the drawbacks:

- Transduction Grammars (J.-C. Raoult)
  - + Still good expressiveness
  - – Not a proper generalization of n-ary rational relations over words
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- Transduction Grammars (J.-C. Raoult)
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  - – Not a proper generalization of *n*-ary rational relations over words
  - – Restriction misses a natural automata theoretic description in our framework
- Separate-Rational Tree Relations (new proposal)
  - \+ Restriction is also natural for asynchronous tree automata
  - \+ Generalize *n*-ary rational word relations
  - – Lack definability of some important relations and classes, e. g. left/right rotations, linear tree transducers

Both generalize automatic tree relations.

Separate-Rational Tree Relations

Definition

The classes SepRat_n of separate-rational tree relations are defined inductively:

- Each finite n-ary tree relation is in SepRat_n
- $R \in \mathrm{SepRat}_n \ \wedge \ |X| = m \ \wedge \ S \in \mathrm{SepRat}_m$
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- Each finite n-ary tree relation is in SepRat_n
  - which only consists of trees with a height of 1 or 2 nodes
  - and has at most one variable of each multivariable in each component.
- $R \in \text{SepRat}_n \ \wedge\ |X| = m \ \wedge\ S \in \text{SepRat}_m$
  $\Longrightarrow\ R \cup S \in \text{SepRat}_n,\ R \cdot_X S \in \text{SepRat}_n,\ R^{*_X} \in \text{SepRat}_n$
  - where each component of a a tuple in R contains at most one variable of X.

Separate-Asynchronous Tree Automata

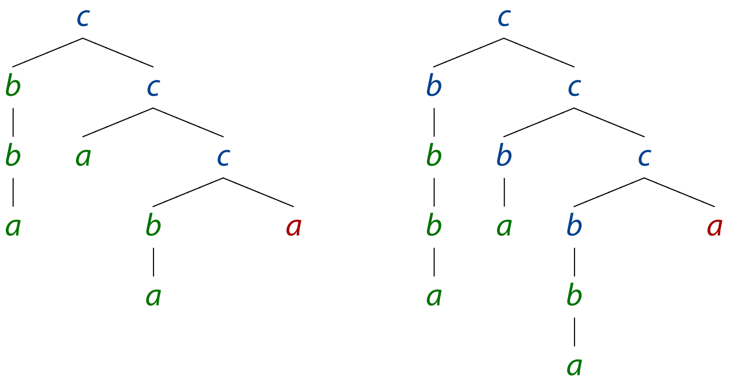
The restriction is also natural for asynchronous tree automata:

- Partition of the state set: Every state q_i of a macro state $(q_1, \ldots, q_m)$ can occur in a run only in one projection.
- Consequence: Synchronization within one projection of the relation is prevented.

Example 3 revisited

Rational expression

$(cx_1y_1, cx_2y_2)^{*_{y_1y_2}} \cdot_{x_1x_2} (x_1, bx_2) \cdot_{y_1y_2} (a, a) \cdot_{x_1x_2} (bx_1, bx_2)^{*_{x_1x_2}} \cdot_{x_1x_2} (a, a)$



Relation is also separate-rational.

Example 2 revisited

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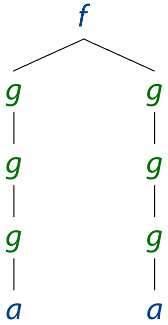
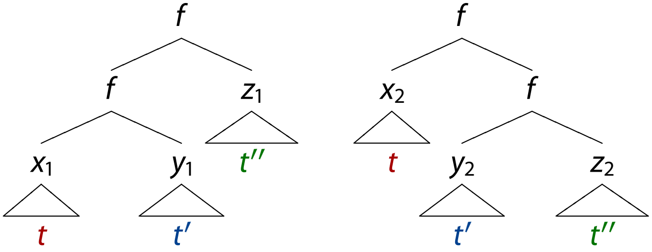


Definable by an rational expression, but not by a separate-rational.

Example 1 revisited

Rational expression

$(ffx_1y_1z_1, fx_2fy_2z_2) \cdot_{x_1x_2} (t, t) \cdot_{y_1y_2} (t', t') \cdot_{z_1z_2} (t'', t'')$



Right rotation is not separate-rational.

Conclusion

- Summary:
  - Rational tree relations are a non-trivial generalization of rational relations over words.
  - Asynchronous tree automata enable further investigation of the theory.
  - Restrictions are required to preserve the good properties of rational word relations.

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  The automata theoretic approach enables the definition of
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- Outlook:

  The automata theoretic approach enables the definition of
  - rational relations over unranked trees
  - deterministic rational tree relations
    - a top-down approach is straightforward
    - the bottom-up approach seems to be challenging