Basic sets in the digital plane

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Outline

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Hilbert's 13th problem

Conjecture (Hilbert's 13th problem, 1900-ies)

Not all continuous functions of several variables are representable as superpositions of continuous functions of less variables.

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... refuted in 1950-ies by V. I. Arnold and A. N. Kolmogorov...

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A set $K \subset \mathbb{R}^n$ is said to be a basic subset of $\mathbb{R}^n$ if for each continuous function $f\colon K \to \mathbb{R}$ there exist continuous functions $g_1, \ldots, g_n\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = g_1(x_1) + \cdots + g_k(x_n)$$

for each $(x_1, \ldots, x_n) \in K$.

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Theorem (Kolmogorov 1957, Ostrand 1965, Sternfeld, 1985)

Let $n \geq 3$. A compactum K can be embedded as a basic subset of $\mathbb{R}^n$ if and only if $\dim K \leq (n-1)/2$.

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Characterize compact spaces basically embeddable into $\mathbb{R}^2$!

Array

Definition (Skopenkov, 1995)

A sequence of (not necessarily different) points $\{a_i\}_{i\in I} \subset \mathbb{R}^2$, where $I = \{1, 2, \ldots, m\}$ or $I = \mathbb{N}$ is an array if

- each $a_i \neq a_{i+1}$ and the segment $[a_i, a_{i+1}]$ is a parallel to one of the coordinate axes
- each two consecutive segments are orthogonal.

If $I = \{1, 2, \ldots, m\}$ then the length is $m - 1$.

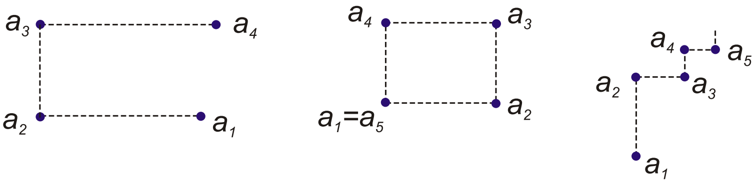

Figure: Array of length 3; closed array; infinite array

Main result – Sternfeld's theorem

Theorem (Sternfeld, 1989, Skopenkov 1995)

A compact set $K \subset \mathbb{R}^2$ is basic, i.e. each continuous $f : K \to \mathbb{R}$ can be decomposed as $f(x, y) = g(x) + h(y)$ for some continuous $g, h : \mathbb{R} \to \mathbb{R}$

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there exists an $m \in \mathbb{N}$ such that K contains no array of length m.

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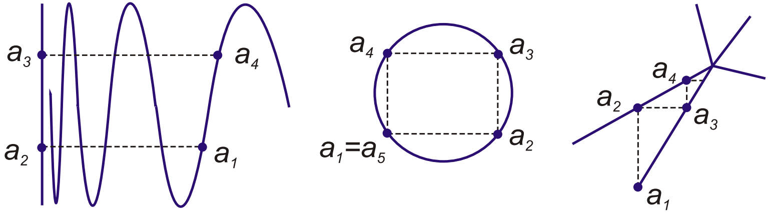

Motivation

Theorem (Neža Mramor Kosta, E.T., 2003, 2007)

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- In each step K is replaced by a finite set of points.
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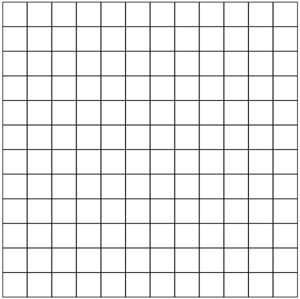
Goals

1. Define the notions in the discrete setting and obtain analogous results.
2. Obtain new results in the digital setting.
3. Find applications.
4. Obtain new results in the continuous setting.

Outline

Introduction



- digital unit square $\mathbb{I}_k^2$ divided into $k \times k$ cells (pixels)
- one cell:

$$e_{ij} = a_i \times b_j = \left[\frac{i-1}{k}, \frac{i}{k}\right) \times \left[\frac{j-1}{k}, \frac{j}{k}\right)$$

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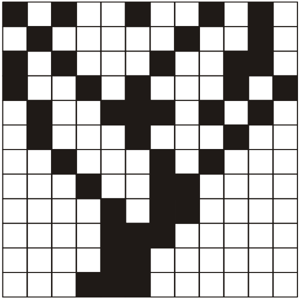

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- $f_k : K_k \to \mathbb{R}$ associates values to e_{ij}

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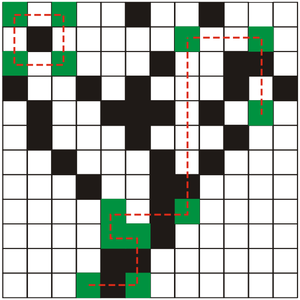

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- $f_k : K_k \to \mathbb{R}$ associates values to e_{ij}
- array is defined analogously

Digital Sternfeld's theorem

Definition

A digital set $K_k \subset \mathbb{I}_k^2$ is basic if for each function $f_k : K_k \to \mathbb{R}$ there exist functions $g_k, h_k : \mathbb{I}_k \to \mathbb{R}$ such that

$$f_k(e_{ij}) = g_k(a_i) + h_k(b_j)$$

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$$\Updownarrow$$

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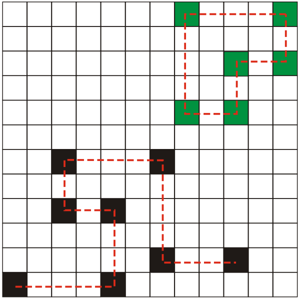
Digital reformulation



- the length of an array in $\mathbb{I}_k^2$ with pairwise different cells is bounded by k^2
- there exists an $m \in \mathbb{N}$ such that K_k contains no array of length $m \Longleftrightarrow K_k$ contains no closed array

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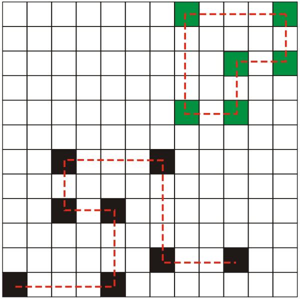

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Theorem (digital reformulation of Sternfeld's theorem)

A digital set $K_k \subset \mathbb{I}_k^2$ is basic, i.e. each $f_k : K_k \to \mathbb{R}$ can be decomposed as $f_k(a_i \times b_j) = g_k(a_i) + h_k(b_j)$ for some $g_k, h_k : \mathbb{I}_k \to \mathbb{R}$

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K_k contains no closed array.

Proof

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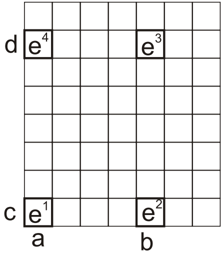
Let K_k contain a closed array $\{e^1, e^2, e^3, e^4, e^5\}$, $e^5 = e^1$:



$$
\begin{aligned}
e^1 &= a \times c \\
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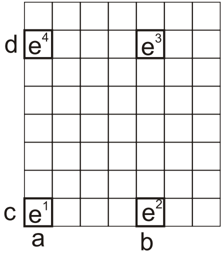

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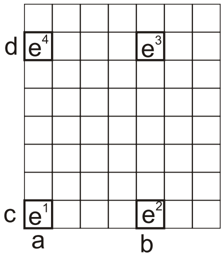

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$$f_k(e^1) - f_k(e^2) + f_k(e^3) - f_k(e^4) = 0$$

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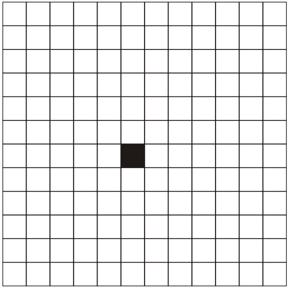

- arbitrary cell $a \times b$
  - $g_k(a)$ arbitrary
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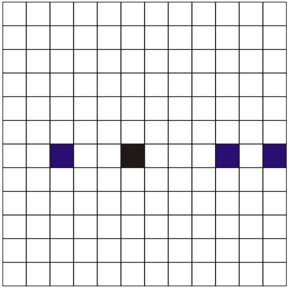

- arbitrary cell $a \times b$
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  - $h_k(b) = f_k(a \times b) - g_k(a)$
- horizontal neighbors of $a \times b$
  - h_k is determined
  - $g_k = f_k - h_k$

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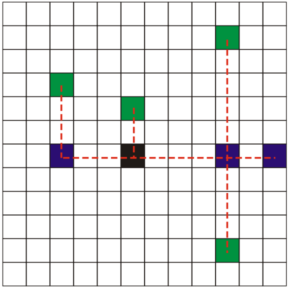

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- vertical neighbors of horizontal neighbors of $a \times b$
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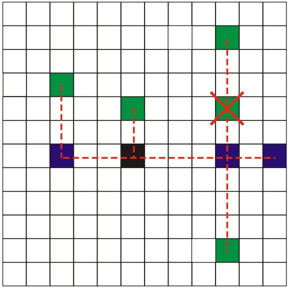

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K_k contains no closed array so there is no conflict.

Basic sets

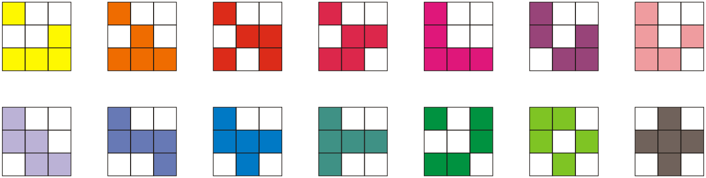
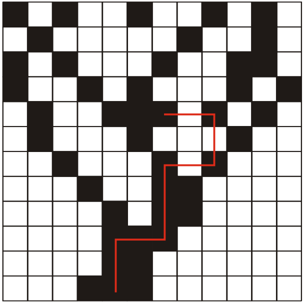


Figure: Maximal basic sets in $\mathbb{I}_3^2$ (up to symmetries of the square)

Cycles in graphs



- Compare finding basic subsets with finding cycles in graphs.
- Vertices are the cells.
- Edges connect cells on the same vertical or horizontal level.
- Arrays correspond to paths.

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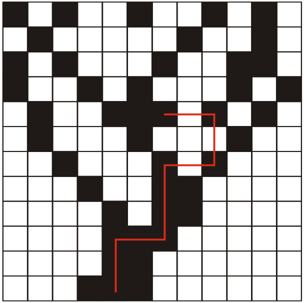

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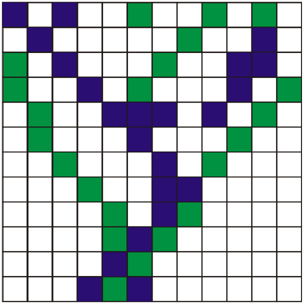

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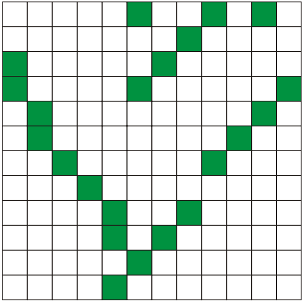

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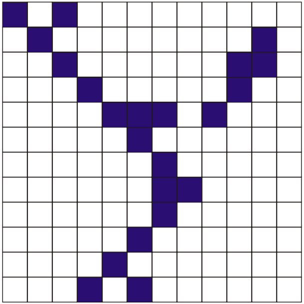

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Digital pictures

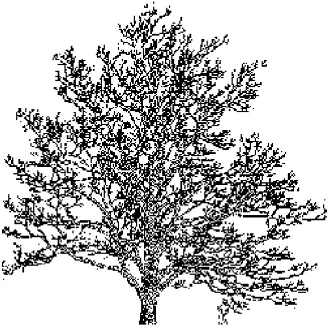
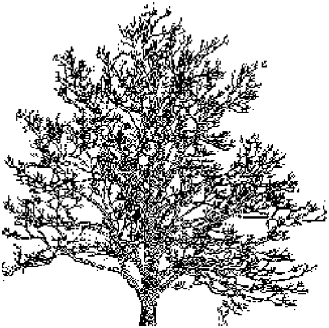


Figure: Original image

Example of a digital function on the digital square is the gray-scale function of a digital image of resolution $k \times k$ (here $k = 249$).

- f_k gives the shades of gray (here white is 0)
- $K_k \subset \mathbb{I}_k^2$ is the set of cells with $f_k \neq 0$

Digital pictures



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- a small number of basic sets may be sufficient for an approximate reconstruction

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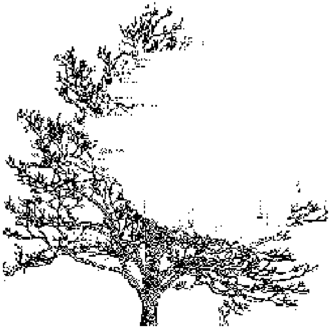

Figure: First 20 basic sets

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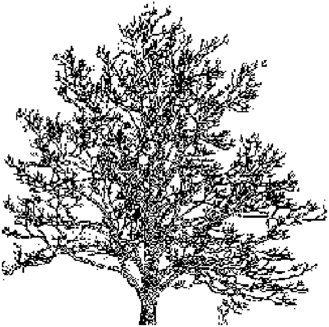

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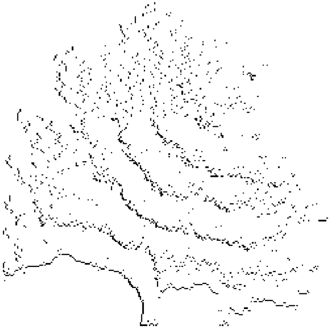

Figure: Every 10th basic set

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Figure: Every 2nd basic set

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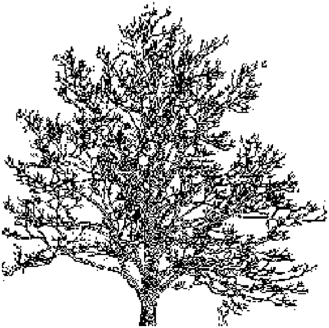

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- $K_k \subset \mathbb{I}_k^2$ is the set of cells with $f_k \neq 0$
- K_k can be decomposed into a union of basic sets (here 61 basic sets)
- a small number of basic sets may be sufficient for an approximate reconstruction

Ideas for further research

Definition

A closed array $\{e^1, \ldots, e^m\} \subset K_k$ is a closed array with respect to $f_k : K_k \to \mathbb{R}$ if

$$f_k(e^1) - f_k(e^2) + f_k(e^3) - \ldots - f_k(e^{m-1}) \neq 0.$$

Theorem

$f_k : K_k \to \mathbb{R}$ can be decomposed as $f_k = g_k + h_k$

$$\Updownarrow$$

K_k contains no closed array with respect to f_k.

Ideas for further research

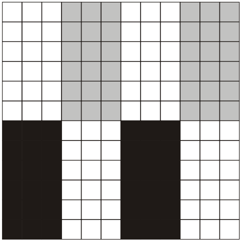

Figure: The gray-scale function can be decomposed

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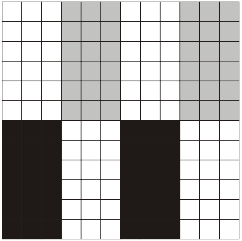

Figure: The gray-scale function can be decomposed

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$$\Updownarrow$$

K_k contains no closed array with respect to f_k.

Definition

A closed array $\{e^1, \dots, e^m\} \subset K_k$ is a closed array with respect to $f_k : K_k \to \mathbb{R}$ and $\varepsilon \geq 0$ if

$$|f_k(e^1) - f_k(e^2) + f_k(e^3) - \dots - f_k(e^{m-1})| \geq \varepsilon.$$

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Thank you for your attention!

Ďakujem za pozornosť!