Assisted Problem Solving and Decompositions of Finite Automata

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Assisted Problem Solving

- we consider the problem of recognizing a formal language
- how can this task be simplified, if we have some a priori information about the input?
- known approaches:

  advice functions - additional information is based on the length of the input

  promise problems - it is promised that inputs come only from some subset of Σ^*

Assisted Problem Solving - Advisors

- we engage an "advisor", that processes the input prior to the "solver"
- solver then obtains some information about the results of the advisor's computation
- we expect that having the information provided by the advisor, the solver's task may become easier
- we also expect the advisor to be simpler than the original solver required for the task, otherwise the advisor would make the task trivial

Assisted Problem Solving and DFAs

- the solver is a DFA trying to recognize some regular language
- the advisor is also a DFA
- we let the solver know some result of the advisor's computation on the input
    - did it accept the input?
    - what was the final state?
- to obtain nontrivial results we require both the advisor and the solver to be simpler than the minimal DFA for the language recognized
- the complexity measure used is the number of states
- this naturally leads to decompositions of finite automata

S.P. Decompositions

- Hartmanis, Stearns - decompositions of sequential machines
- central concept: S.P. partition

Definition

A partition π on the set of states of a sequential machine M is an S.P. partition, if

$$p \equiv_\pi q \Rightarrow (\forall a \in \Sigma; \delta(p,a) \equiv_\pi \delta(q,a))$$

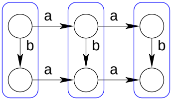

Figure: S.P. partition

- join and meet can be defined on S.P. partitions of M, they form a finite lattice (sublattice of the lattice of all partitions on the set of states of M)

Parallel Decomposition of State Behavior

Theorem (Hartmanis, Stearns)

A sequential machine M has a parallel decomposition of state behavior iff there exist S.P. partitions π_1, π_2 on the set of its states, such that $\pi_1 \cdot \pi_2 = 0$.

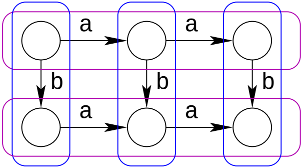

Figure: S.P. partitions π_1 and π_2, such that $\pi_1 \cdot \pi_2 = 0$.

New Types of Decomposition

- based on our motivation, we define new DFA decompositions
- what should the results of independent computations of both automata forming the decomposition say about the result of the computation of the original automaton?

Definition

Decomposition of a DFA A into simpler DFAs A_1 and A_2:

SI-decomposition – knowing the final states of A_1 and A_2, we can determine the final state of A

AI-decomposition – knowing whether A_1 and A_2 accept, we can determine whether A accepts

wAI-decomposition – knowing the final states of A_1 and A_2, we can determine whether A accepts

Decomposition of State Behavior

- adaptation of the notion from [Hartmanis, Stearns] for the DFA setting without loosing the connection to the useful concept of S.P. partitions
- to be related to our new definitions

Definition

(A_1, A_2) forms an SB-decomposition of A, if $\exists \alpha : K \xrightarrow{inj.} K_1 \times K_2$

(i) $(\forall a \in \Sigma)(\forall q \in K); \alpha(\delta(q, a)) = (\delta_1(q_1, a), \delta_2(q_2, a))$ where $\alpha(q) = (q_1, q_2)$

(ii) $\alpha(q_0) = (q_0^{(1)}, q_0^{(2)})$

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and forms an ASB-decomposition, if moreover

(iii) $(\forall q \in K); \alpha(q) \in F_1 \times F_2 \Leftrightarrow q \in F$.

Existence of the SB-decomposition

- the result of [Hartmanis, Stearns] holds also in the DFA setting

Theorem

A DFA A has an SB-decomposition iff there exist S.P. partitions π_1, π_2 on the set of its states, such that $\pi_1 \cdot \pi_2 = 0$.

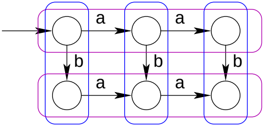


Figure: S.P. partitions π_1 and π_2, such that $\pi_1 \cdot \pi_2 = 0$.

Existence of the ASB-decomposition

- compared to SB-decompositions, we also have to take accepting states into account
- when deciding based on accepting behavior of A_1 and A_2, we do not know, which of the accepting states was final, hence all possible pairs must lead to the same behavior in A

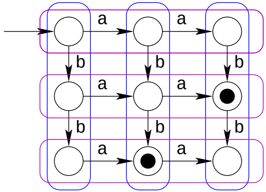


Figure: S.P. partitions π_1 and π_2 that do not induce an ASB-decomposition.

Existence of the ASB-decomposition (contd.)

Definition

Partitions $\pi_1 = \{R_1, \ldots, R_k\}$ and $\pi_2 = \{S_1, \ldots, S_l\}$ on the set of states of a DFA A separate the final states, if there exist indices $i_1, \ldots, i_r, j_1, \ldots, j_s$, such that $(R_{i_1} \cup \ldots \cup R_{i_r}) \cap (S_{j_1} \cup \ldots \cup S_{j_s}) = F$.

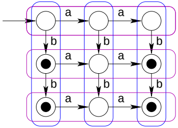


Figure: π_1 and π_2 separate the final states.

Theorem

DFA A has an ASB-decomposition iff there exist S.P. partitions π_1 and π_2 on the set of its states, such that they separate the final states and it holds $\pi_1 \cdot \pi_2 = 0$.

Existence of the AI-decomposition and wAI-decomposition

- similar technique yields the following sufficient conditions for the existence of the AI-decomposition and wAI-decomposition:

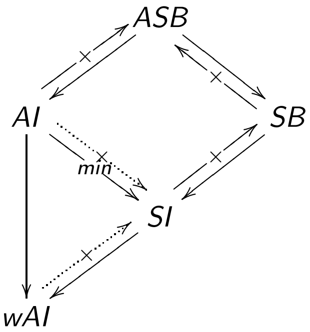
Theorem

Let A be a DFA and let π_1 and π_2 be S.P. partitions on the set of its states, such that they separate the final states of A. Then A has an AI-decomposition.

Theorem

Let A be a DFA and let π_1 and π_2 be S.P. partitions on the set of its states, such that it holds $\pi_1 \cdot \pi_2 \preceq \{F, K - F\}$. Then A has an wAI-decomposition.

Relations between types of decomposition



$A \longrightarrow B$: every A-decomposition is also a B-decomposition

$A \xrightarrow{min} B$: every A-decomposition of a minimal DFA is also a B-decomposition

$A \overset{\times}{\dashrightarrow} B$: not every A-decomposition is also a B-decomposition

$A \overset{\times}{\longrightarrow} B$: there exists a DFA that has a nontrivial A-decomposition but does not have a nontrivial B-decomposition

Perfectly Decomposable Automata

(A_1, A_2) is a perfect decomposition of A, if $|K_1| \cdot |K_2| = |K|$.

$$\begin{tikzcd} AI = ASB \arrow[d] \\ SI = SB \arrow[d] \\ wAI \end{tikzcd}$$

- for DFAs without unreachable states, each perfect SI-decomposition is also a perfect SB-decomposition
- for minimal DFAs, each perfect AI-decomposition is also a perfect ASB-decomposition
- hence, we can use the derived necessary and sufficient conditions for the existence of ASB- and SB-decompositions to decide the existence of perfect AI- a SI-decompositions

Decomposability of the Minimal DFA

- How is the decomposability of a DFA related to the decomposability of the corresponding minimal automaton?

Theorem

Let A be a DFA and A_{min} be the minimal DFA equivalent to A. Then

1. *If (A_1, A_2) form an AI-decomposition (SI-decomposition, wAI-decomposition) A, then they form also a decomposition of A_{min} of the same type.*
2. *If (A_1, A_2) form an SB-decomposition A and the S.P.-lattice of DFA A is distributive, then there also exists an SB-decomposition of A_{min}, such that its i^{th} DFA has at most as many states as A_i, $i \in \{1, 2\}$.*

Decomposability of the Minimal DFA (contd.)

- without distributivity the last claim does not hold
- consider the language $L = \{a^{2k}b^{2l} \mid k \geq 0, l \geq 1\}$
- the minimal automaton is SB-undecomposable
- a non-minimal automaton can be ASB-decomposed into two, both having less states than the minimal one

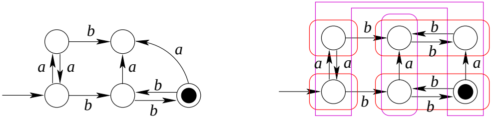


Figure: The minimal and the decomposable automaton for L.

Achievable Degrees of Decomposability

- we know that undecomposable and perfectly decomposable automata exist
- what is the situation between these two extreme points?

Theorem

Let $n \in \mathbb{N}$ be such that $n = k + r \cdot s$, where $r, s, k \in \mathbb{N}$, $r, s \geq 2$. Then there exists a minimal DFA A consisting of n states, such that it has only one nontrivial nonredundant SB-decomposition (ASB-decomposition) up to the order of the automata in the decomposition, and this decomposition consists of automata with $k + r$ and $k + s$ states.

Sketch of the Proof

Lemma

For each $r, s \in \mathbb{N}$, $r, s \geq 2$, there exists a minimal DFA A consisting of $r \cdot s$ states and having only one nontrivial nonredundant SB-decomposition (ASB-decomposition) up to the order of automata, consisting of automata having r and s states.

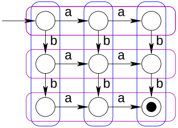


Figure: An example for $r = s = 3$.

Sketch of the Proof (contd.)

Lemma

Let A be a DFA consisting of n reachable states. Let A' be its k-extension. Then A has a nontrivial nonredundant SB-decomposition (ASB-decomposition) consisting of automata having r and s states iff A' has a nontrivial nonredundant decomposition of the same type, consisting of automata having $k + r$ and $k + s$ states.

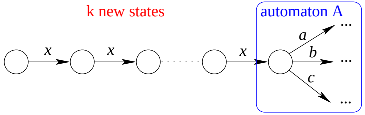


Figure: The k-extension of an automaton A.

Summary

Our contribution:

- initiating the study of assisted problem solving
- applying it to the world of finite automata
  - defining new DFA decompositions modelling this intuitive concept
  - deriving conditions for the existence of these decompositions
  - inspecting the notion of perfect decomposability
  - exploring the decomposability of the corresponding minimal DFA
  - inspecting the various degrees of decomposability that can be achieved

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Thank you for your attention.