Periodic and Infinite Traces in Matrix Semigroups

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Overview

- Matrix Semigroups Problems
- Post Correspondence Problem (PCP)
- Fixed Element PCP (FEPCP)
- Application of FEPCP
  - *Any Diagonal Matrix Problem,*
  - *Recurrent Matrix Problem*
  - *Vector Ambiguity Problem*
  - *Infinite Post Correspondence Problem*

Matrix Semigroup Problems

- Given a set of matrices: $G=\{M_1, M_2,\dots, M_k\}$
  - **(Membership problem)**

    Can we construct a product of matrices that is equal M?
  - **(Vector Reachability problem)**

    Given a pair of vectors x,y. Whether exist M in S: Mx=y?
  - **(Freeness problem)**

    Can we check that a generator G forms a free multiplicative group of matrices?

  - We want to study more structural properties about matrix semigroups: *Any Diagonal Matrix Problem, Recurrent Matrix Problem, Vector Ambiguity Problem*

Problems about periodic and infinite traces

- ***Any Diagonal Matrix Problem***

  Given a finite set of matrices G generating a semigroup S. Does there exist any matrix D $\in$ S such that D is a diagonal matrix?

- ***Recurrent Matrix Problem***

  Given a matrix B and a semigroup S generated by a finite set of matrices G. Does B have an infinite number of factorizations over elements of *G?*

- **Vector Ambiguity Problem**

  Given a semigroup S of (n x n)-matrices and an initial n-dimensional vector u. Let V be a set of vectors such that

  $V = \{v : v = Mu; M \in S\}$. Do S and u generate a non-repetitive set of vectors?

  In other words the question is whether for every vector v of set V there is a unique matrix $M \in S$ such that $M \cdot u = v$?

Post's Correspondence Problem

We are given an indexed set of pairs of (binary) words $\{(u_1, v_1), (u_2, v_2), \cdots, (u_n, v_n)\}$.

PCP(n): Does there exist a finite sequence $(i_1, i_2, \ldots, i_k)$ such that $u_{i_1} u_{i_2} \cdots u_{i_k} = v_{i_1} v_{i_2} \cdots v_{i_k}$?

$$P_1 = \left[\frac{aab}{a}\right], P_2 = \left[\frac{ba}{a}\right], P_3 = \left[\frac{ab}{bbaabb}\right]$$

Now take the sequence $P_1 \cdot P_2 \cdot P_3 \cdot P_2$:

$$\begin{array}{cccc} aab & ba & ab & ba \\ a & a & bbaabb & a \end{array}$$

It is known that PCP(2) is decidable and PCP(7) is undecidable.

Fixed Element PCP (direct reachability)

Fixed Element PCP problem is undecidable for 14 pairs of words.

Given an alphabet $\Gamma=\{a,a^{-1},b,b^{-1},c,c^{-1},*\}$ and a finite set of pairs of words over Γ, $P = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subset \Gamma^* \times \Gamma^*$.

Does there exist a finite sequence of indices $s = (s1, s2, \ldots, sk)$ *such that*

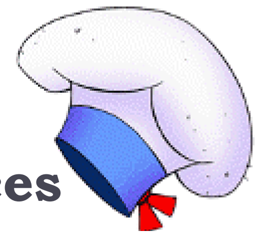
$$u_{s1}u_{s2} \cdots u_{sk} = v_{s1}v_{s2} \cdots v_{sk} = * \quad ?$$

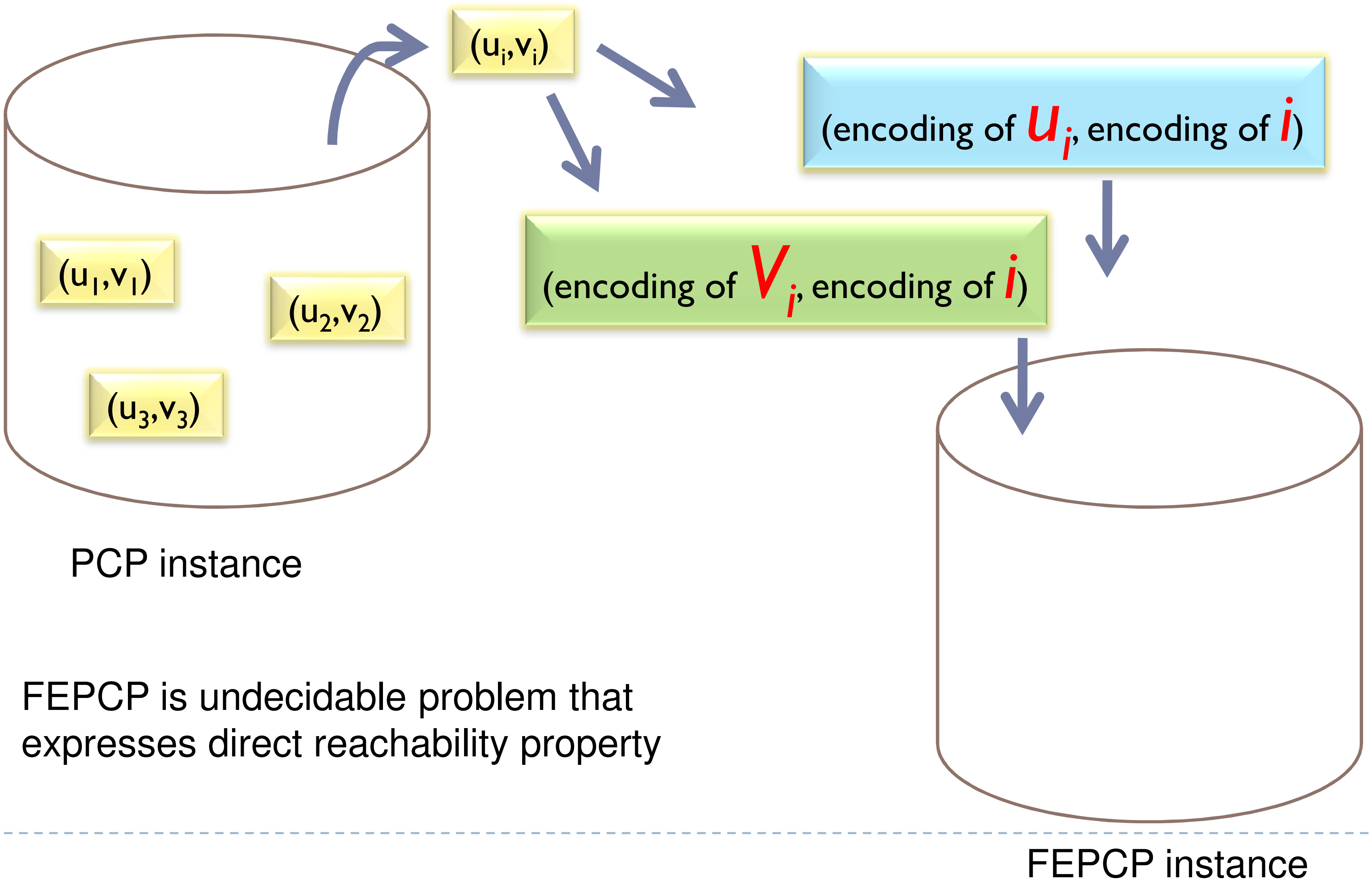
If we could fix that one pair (for example with an index s_1) will be used exactly once then we can have pairs from aphabet $\{a,a^{-1},b,b^{-1},c,c^{-1}\}$ and undecidability problem about whether exist a finite sequence of indices $s = (s1, s2, \ldots, sk)$ *such that*

$$u_{s1}u_{s2} \cdots u_{sk} = v_{s1}v_{s2} \cdots v_{sk} = \varepsilon \quad ?$$

Recipe for Fixed Elelment PCP

Ingredients: **PCP instances and encoding of indices**





FEPCP is undecidable problem that expresses direct reachability property

Tiling in FEPCP (basic idea)

$u_1 = S_1 S_2 S_3 S_4 \quad S_1 S_2 = v_1$

$u_2 = S_5 S_6 S_7 \quad S_3 S_4 S_5 S_6 = v_2$

$u_3 = S_8 \quad S_7 S_8 = v_3$

$S_1 S_2 S_3 S_4 \; S_5 S_6 S_7 \; S_8$

$S_1 S_2 \; S_3 S_4 S_5 S_6 \; S_7 S_8$

$$\varphi(u^{\mathrm{R}}) \times \psi(v)$$

$S_8 \; S_7 S_6 S_5 \; S_4 S_3 S_2 S_1 \; S_1 S_2 \; S_3 S_4 S_5 S_6 \; S_7 S_8$

$i \mapsto 0^i 1$

- We use a special index coding which also forms a palindrome:

(1) 01000101001 $\overline{\text{(1) 00101000101}}$

1 3 1 2 2 1 3 1

Applications of FEPCP

- ***Any Diagonal Matrix Problem***

  Given a finite set of matrices G generating a semigroup S. Does there exist any matrix $D \in S$ such that D is a diagonal matrix?

- ***Recurrent Matrix Problem***

  Given a matrix B and a semigroup S generated by a finite set of matrices G. Does B have an infinite number of factorizations over elements of *G?*

- **Vector Ambiguity Problem**

  Given a semigroup S of $(n \times n)$-matrices and an initial n-dimensional vector u. Let V be a set of vectors such that

  $V = \{v : v = Mu; M \in S\}$. Do S and u generate a non-repetitive set of vectors?

  In other words the question is whether for every vector v of set V there is a unique matrix $M \in S$ such that $M \cdot u = v$?

Any Diagonal Matrix Problem

Given a finite set of matrices G generating a semigroup S. Does there exist any matrix D ∈ S such that D is a diagonal matrix?

$$\zeta(a) = \begin{pmatrix} \frac{3}{5}+\frac{4}{5}\mathbf{i} & 0 \\ 0 & \frac{3}{5}-\frac{4}{5}\mathbf{i} \end{pmatrix}, \zeta(b) = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\zeta(\overline{a}) = \begin{pmatrix} \frac{3}{5}-\frac{4}{5}\mathbf{i} & 0 \\ 0 & \frac{3}{5}+\frac{4}{5}\mathbf{i} \end{pmatrix}, \zeta(\overline{b}) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$$

$$\gamma(\star) = \zeta(a),\ \gamma(a) = \zeta(bab),\ \gamma(b) = \zeta(b^2a^2b^2),\ \gamma(\varDelta) = \zeta(b^3a^3b^3)$$
$$\gamma(\overline{a}) = \zeta(\overline{b}\overline{a}\overline{b}),\ \gamma(\overline{b}) = \zeta(\overline{b}^2\overline{a}^2\overline{b}^2),\ \gamma(\overline{\varDelta}) = \zeta(\overline{b}^3\overline{a}^3\overline{b}^3),$$

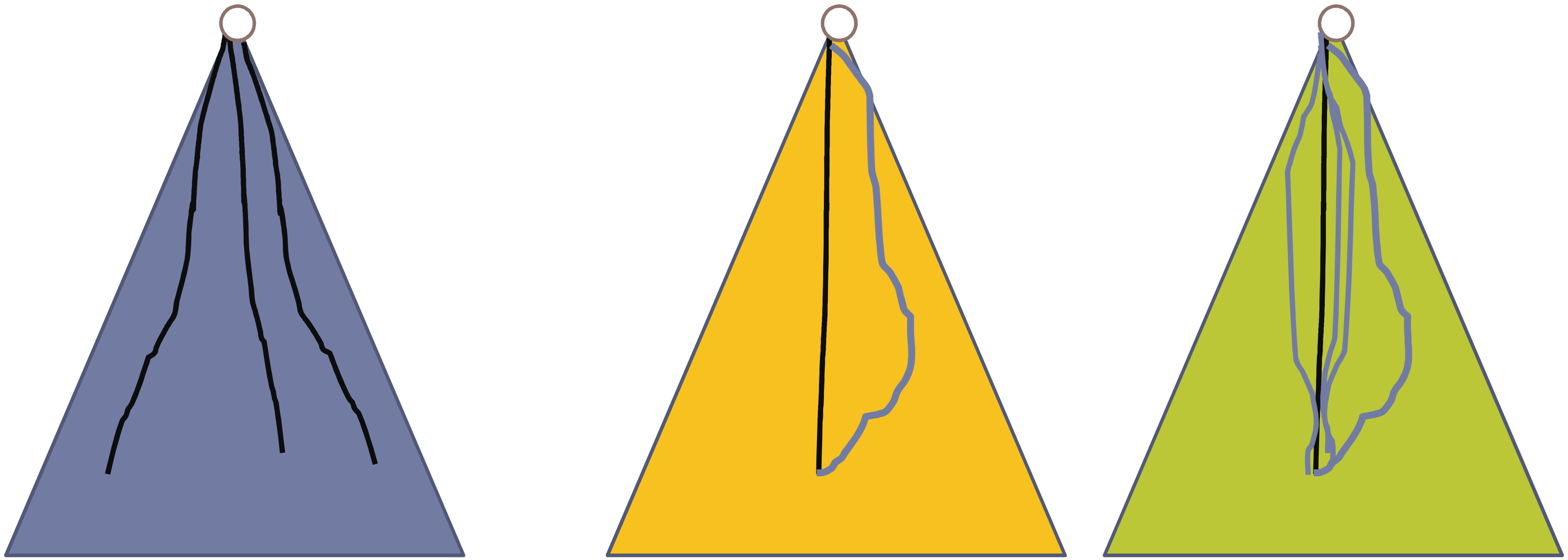
We create a final set of 4-dimensional rational complex matrices:

$$T = \{\gamma(u_1)\otimes\gamma(v_1), \gamma(u_2)\otimes\gamma(v_2), \ldots, \gamma(u_n)\otimes\gamma(v_n)\} \subseteq \mathbb{C}(\mathbb{Q})^{4\times 4}.$$

$$D = \gamma(u_{w_1}u_{w_2}\cdots u_{w_k})\otimes\gamma(v_{w_1}v_{w_2}\cdots v_{w_k}) = \gamma(\star)\otimes\gamma(\star) = \zeta(a)\otimes\zeta(a),$$

More questions about freeness related properties

- Matrix semigroup is free iff every product of basis matrices provides a unique transformation.



Free matrix semigroup

Non-Free matrix semigroups

Can be non-free with a constant number or with an infinite number of equivalent paths

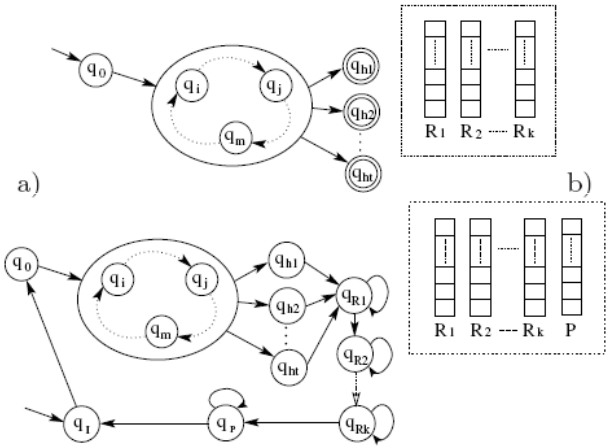
Recurrent Matrix Problem

is undecidable for 4x4 matrix semigroup over Z and 3x3 matrices over Q

Given a matrix B and a semigroup S generated by a finite set of matrices G. Does B have an infinite number of factorizations over elements of *G?*

There are two ways to prove:

1. **via simulation of special form of Minsky machine**
   Gives just a proof of above fact
2. **via simulation of lossy FIFO automata**
   We can define a class of matrix semigroups with decidable membership and undecidable Recurrent Matrix Problem



Minsky Machine

Lossy FIFO automata

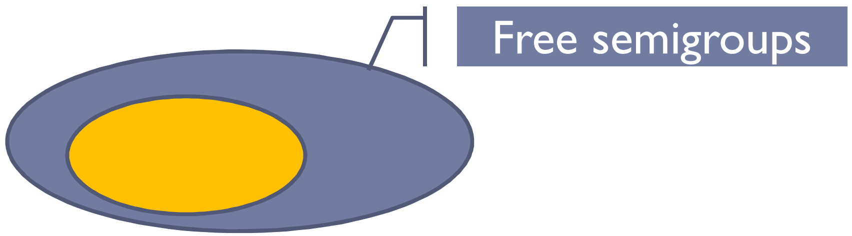
Interpretation of computations in terms of concatination of pair of words (PCP style)

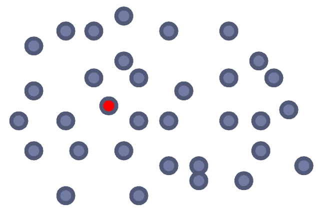
Application of FEPCP

Vector Ambiguity Problem

Free matrix semigroup provides the opportunity to construct a unique linear transformation by any combination of basis matrices.

Not every free semigroup can generate a non-repetitive set of vectors?





Given a semigroup S of (n x n)-matrices and an initial n-dimensional vector u. Let V be a set of vectors such that $V = \{v : v = Mu; M \in S\}$*. It is undecidable to check whether S and u generate a non-repetitive set of vectors.*

Proof idea: Simulate Minsky machine by PCP type of computation, encode it in terms of matrix multiplications. Only correct simulation my lead to the vector ambiguity, which is possible if there exist a periodic trajectory in MM.

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