Mortality Problem for 2 $\times$ 2 Integer Matrices

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The Mortality Problem

$\mathbb{K} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$

$M(n, \mathbb{K})$ set of $n \times n$ matrices with entries in $\mathbb{K}$

Mortality problem [$Mort_{\mathbb{K}}(n)$]

- Instance: a finite set $F \subseteq M(n, \mathbb{K})$.
- Question: $\underline{0} \in \langle F \rangle$?

- The mortality problem is decidable for any set of row-monomial matrices over $\mathbb{Q}$ (A. Lisitsa, I. Potapov).
- $Mort_{\mathbb{Z}}(3)$ is undecidable (M.S. Paterson).
- **Open problem**: is $Mort_{\mathbb{Z}}(2)$ decidable?

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- **Open problem**: is $Mort_{\mathbb{Z}}(2)$ decidable?

Reduction of $Mort_{\mathbb{Z}}(2)$ to Vector Reachability Problem

Let $F = \{S_1 \ldots, S_k, R_1, \ldots, R_n\}$, where R_i are invertible and S_j are not invertible.

Lemma (O. Bournez, M. Branicky)

$\underline{0} \in \langle F \rangle$ iff there exist $i, j \in \{1, \ldots, k\}$, $K \in \langle R_1, \ldots, R_n \rangle$ such that

$$S_j K S_i = \underline{0}.$$

- $\ker(S_i) = \langle k_i \rangle$, $Im(S_i) = \langle \iota_i \rangle$

Reduced problem

Check if there exist $i, j \in \{1, \ldots, k\}$, $K \in \langle R_1, \ldots, R_n \rangle$ such that

$$K \iota_i^T \in ker(S_j).$$

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Our Contribution

- Considered case: $\{R_1, \ldots, R_n\} \subseteq GL(2, \mathbb{Z})$.
- We prove that in this case $Mort_{\mathbb{Z}}(2)$ is decidable.

Reduction (1)

Remark

- $\iota_i = (x, y)$, $gcd(x, y) = 1$
- $k_j = (x', y')$, $gcd(x', y') = 1$

From this fact and from $K \in GL(2, \mathbb{Z})$ it follows

Lemma

Let $K \in \langle R_1, \ldots, R_n \rangle$, $i, j \in \{1, \ldots, k\}$. Then

$$K\iota_i^T \in \ker(S_j) \text{ iff } K\iota_i^T = \pm k_j^T.$$

Reduction (2)

Remark

- Let $\iota_i = (x, y) \in \mathbb{Z}^2$ with $gcd(x, y) = 1$. Then there exists $U \in GL(2, \mathbb{Z})$ such that $U(1, 0)^T = \iota_i^T$.
- Checking if there exists $K \in \langle R_1, \ldots, R_n \rangle$ such that $K\iota_i^T = \pm k_j^T$ is equivalent to the problem of checking whether there exists $U^{-1}KU \in \langle U^{-1}R_1U, \ldots, U^{-1}R_nU \rangle$ such that $U^{-1}KU(1, 0)^T = \pm U^{-1}k_j^T$.

Reduced problem

Does there exist $K' \in \langle R'_1, \ldots, R'_n \rangle$ such that

$$K'(1, 0)^T = \pm(a, b)^T$$

with $gcd(a, b) = 1$?

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Reduction (3)

Proposition

Let $a, b \in \mathbb{Z}$ be relative prime. Then $K \in GL(2, \mathbb{Z})$ maps $(1, 0)^T$ to $(a, b)^T$ if and only if K is of the form AT^λ where $\lambda \in \mathbb{Z}$,

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix}$$

with c, d integers satisfying $ad - bc = 1$.

Reduced problem

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$$AT^\lambda \in \langle R'_1, \ldots, R'_n \rangle ?$$

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Our main result

- $M \simeq \Sigma^* / \equiv$ finitely presented monoid.
- Q is a **rational subset** of M if there exists a finite automaton $\mathcal{A}$ with input alphabet Σ such that $x \in Q$ if and only if there exists a word w in the language recognized by $\mathcal{A}$ with $x = [w]_\equiv$.

Theorem

Given a rational subset Q of matrices in $M(2, \mathbb{Z})$ and a matrix $A \in GL(2, \mathbb{Z})$, it is recursively decidable whether or not

$$\{AT^m \mid m \in \mathbb{Z}\} \cap Q \neq \emptyset$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof of Theorem 1/4

Let $\mathcal{R}(X_1, \ldots, X_n)$ be a rational expression such that $Q = \varphi(\mathcal{R}(X_1, \ldots, X_n))$, where φ assigns a matrix to each symbol X_i.
We have to check whether or not

$$\{AT^m \mid m \in \mathbb{Z}\} \cap \varphi(\mathcal{R}(X_1, \ldots, X_n)) = \emptyset \tag{1}$$

- **Claim 1:** $A = I$.
  If X is a new symbol and we extend φ so that $\varphi(X) = A^{-1}$, then the condition (1) is equivalent to the condition

$$\{T^m \mid m \in \mathbb{Z}\} \cap \varphi(X \cdot \mathcal{R}(X_1, \ldots, X_n)) = \emptyset.$$

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Proof of Theorem 2/4

We have to check whether or not

$$\{T^m \mid m \in \mathbb{Z}\} \cap \varphi(\mathcal{R}(X_1, \ldots, X_n)) = \emptyset \tag{2}$$

- **Claim 2:** the determinant of all X_i's is equal to ± 1 (since $det(T^m) = 1$).
- **Claim 3:** the determinant of all X_i's is equal to 1.

So we can suppose that $\varphi(\mathcal{R}(X_1, \ldots, X_n)) \subseteq SL(2, \mathbb{Z})$.

Intersection Of Rational Subsets

$$SL(2,\mathbb{Z})/\{I,-I\} = PSL(2,\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z},$$

Proposition

Let G be a group such that $G \simeq \mathbb{Z}/p_1\mathbb{Z} * \ldots * \mathbb{Z}/p_n\mathbb{Z}$. Let $\mathcal{A}, \mathcal{B}$ be two automata that recognize respectively two rational subsets $H, K \subseteq G$. Then it is recursively decidable whether or not $H \cap K \neq \emptyset$.

So we can verify whether the image of

$$\{T^m \mid m \in \mathbb{Z}\} \cap \varphi(\mathcal{R})$$

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Proof of Theorem 3/4

$$T^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

Let $\psi : SL(2, \mathbb{Z}) \rightarrow M(2, \mathbb{Z}/3\mathbb{Z})$ be the morphism which puts the entries of a matrix module 3. Then

$$\{T^m \mid m \in \mathbb{Z}\} \subseteq H =$$
$$= \left\{ A \in SL(2, \mathbb{Z}) \mid \psi(A) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \text{ mod } 3, i = 0, 1, 2 \right\}$$

while $-T^m \notin H$.

Proof of Theorem 4/4

The set

$$P = \left\{ W \in \{X_1, \ldots, X_n\}^* | \psi(\varphi(W)) = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \text{ mod } 3, \right.$$
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is rational.

$$\{T^m \mid m \in \mathbb{Z}\} \cap \varphi(\mathcal{R}(X_1, \ldots, X_n)) = \emptyset$$

iff

$$\{T^m \mid m \in \mathbb{Z}\} \cap (\varphi(\mathcal{R}(X_1, \ldots, X_n) \cap P)) = \emptyset.$$

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Open problem

The mortality problem is decidable for any set of $n \times n$ integer matrices whose determinants are $0, \pm 1$?