Recursive domain equations of filter models (SOFSEM ’08)

Fabio Alessi and Paula Severi

Dipartimento di Matematica e Informatica

Università di Udine

via delle Scienze 208 - Udine (ITALY)

alessi@dimi.uniud.it

ps56@mcs.le.ac.uk

Overview of the talk

- The setting: the Semantics of untyped λ-calculus in the category of ω-algebraic lattices.
- Focus on the *filter models*: definitions, some history.
- The investigation of the paper: the relation between colimits of functors and filter models.
- The classification result on *disciplined* filter models.

The general setting

Classic Untyped λ-Calculus Semantics is developed in various categories:

- Partial orders/Lattices
- di-Domains
- Qualitative Domains
- Quantitative Domains
- Game Semantics
- ...

λ-Calculus Semantics in ALG

ALG: the category of ω-algebraic lattices and Scott continuous functions. Three main kind of λ-models:

Graph models

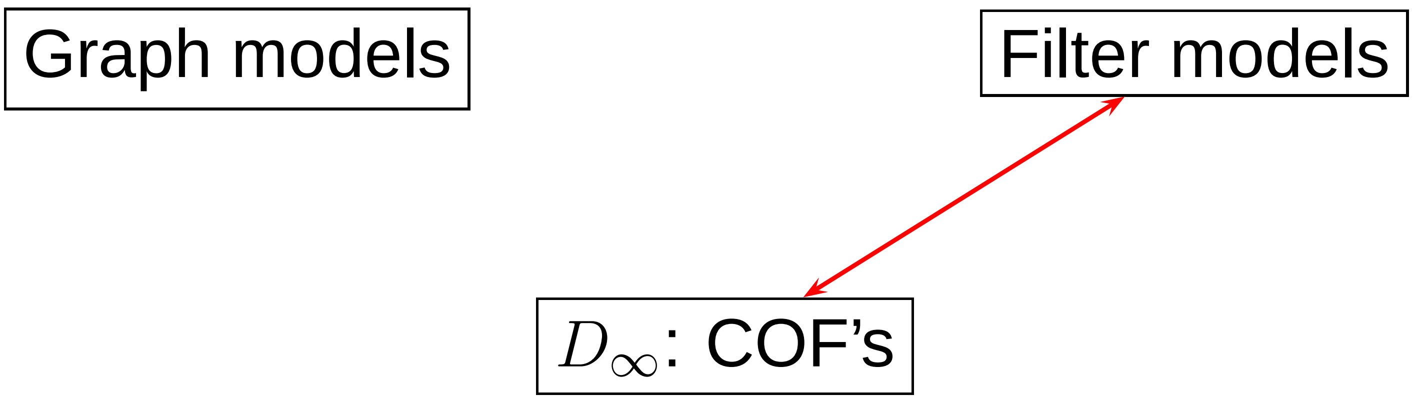
Filter models

D_∞: COF's

COF = colimit of non-trivial ($\neq$ *Id*) continuous functor in **ALG**: Fix a continuous functor H : **ALG** $\rightarrow$ **ALG** and solve the equation $X = H(X)$ in **ALG**^{ep} starting by some initial D_0 and $i_0 : D_0 \rightarrow H(D_0)$ (typically the equation solved is $X = [X \rightarrow X]$)

λ-Calculus Semantics in ALG

ALG: the category of ω-algebraic lattices and Scott continuous functions. Three main kind of λ-models:



COF = colimit of non-trivial ($\neq$ *Id*) continuous functor in **ALG**:
Fix a continuous functor H : **ALG** $\rightarrow$ **ALG** and solve the equation $X = H(X)$ in **ALG**^{ep} starting by some initial D_0 and $i_0 : D_0 \rightarrow H(D_0)$ (typically the equation solved is $X = [X \rightarrow X]$)

The structures

All three kinds of λ-models are structures $\langle D, F, G \rangle$, where $(D, \sqsubseteq)$ ω-algebraic lattice, and D satisfies the reflexivity property $[D \to D] \trianglelefteq D$:

$$D \underset{G}{\overset{F}{\rightleftarrows}} [D \to D]$$

such that $F \circ G = Id_{[D \to D]}$

Reflexivity of D implies that D is a λ-model.

D has arbitrary meets ($d \sqcap e$ or $\sqcap_{i \in I} d_i$) and joins ($d \sqcup e$ or $\bigsqcup_{i \in I} d_i$) and a *countable basis* of compact elements. $[D \to D]$ is the lattice of Scott continuous functions $f : D \to D$ such that $f(\bigsqcup Z) = \bigsqcup f(Z)$ for any *directed* $Z \subseteq D$.

Interpreting λ-terms

- Term interpretation in $\langle D, F, G\rangle$:
  - $[\![x]\!]_\rho = \rho(x)$ (ρ : Var $\longrightarrow D$)
  - $[\![MN]\!]_\rho = F([\![M]\!]_\rho)([\![N]\!]_\rho)$
  - $[\![\lambda x.M]\!]_\rho = G(X \mapsto [\![M]\!]_{\rho[x/X]})$
- If $F \circ G =$ *Id* then $\langle D, F, G\rangle$ is a λ-*model* i.e. (roughly!): terms converted according to β-reduction rule have the same interpretation:

$$[\![(\lambda x.M)N]\!]_\rho = [\![M]\!]_{\rho(x/[\![N]\!]_\rho)}$$

- λ-terms: $M ::= x \mid (MM) \mid (\lambda x.M)$ ($x \in$ Var)

- (β) $(\lambda x.M)N \rightarrow M[x/N]$

Building reflexive structures

- Set-theoretically: graph models ($\mathcal{P}\omega$, $\mathcal{T}$, Engeler model);
- Using *intersection type theories*: filter models;
- By *categorical construction*: COF's solving suitable domain equations $X = F(X)$ in **ALG** (Scott, Park D_∞, Abramsky model for lazy lambda calculus).

Intersection types and filter models

Intersection type theories: they consist of

1. Type Language Π^{∇}: $A = \Omega \mid \varphi \mid A \to A \mid A \cap A \quad (\varphi \in C)$
2. Preorder relation $\leq$ on Π^{∇} that contains:

   (refl) $A \leq A$

   (idem) $A \leq A \cap A$

   (incl$_L$) $A \cap B \leq A$

   (incl$_R$) $A \cap B \leq B$

   (mon) $\dfrac{A \leq A' \quad B \leq B'}{A \cap B \leq A' \cap B'}$

   (trans) $\dfrac{A \leq B \quad B \leq C}{A \leq C}$

   (Ω) $A \leq \Omega$

   (Ω-*lazy*) $A \to B \leq \Omega \to \Omega$

   ($\to$-$\cap$) $(A \to B) \cap (A \to C) \leq A \to (B \cap C)$

   (η) $\dfrac{A' \leq A \quad B \leq B'}{A \to B \leq A \to B'}$

3. Intersection type theory Σ^{∇}: it consists of all the derivable judgments $A \leq B$.

The Filter Structure over Σ^{∇}

- A subset $X \subseteq \Pi^{\nabla}$ is a *filter* over Σ^{∇} iff:
  - is non-empty: $\Omega \in X$
  - is upward closed: $A \leq B$ and $A \in X$ implies $B \in X$;
  - is closed under intersection: if $A, B \in X$, then $A \cap B \in X$
- $\mathcal{F}^{\nabla}$ is the set of filters on Σ^{∇}, ordered by set-theoretic inclusion. ($\uparrow Z$ is the filter generated by Z, for each Z.)
- $F^{\nabla} : \mathcal{F}^{\nabla} \to [\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}]$ and $G^{\nabla} : [\mathcal{F}^{\nabla} \to \mathcal{F}^{\nabla}] \to \mathcal{F}^{\nabla}$ are defined:

$$F^{\nabla}(X) = \quad Y \mapsto \{B \mid \exists A \in Y. A \to B \in X\}$$
$$G^{\nabla}(f) = \uparrow \{A \to B \mid B \in f(\uparrow A)\}$$

- In general $\mathcal{F}^{\nabla}$ is NOT a λ-model, BUT looking for special purpose λ-models is easy using filter models.

The Type Assignment System (TAS):

Γ finite set of premises $x : A$

(Ax) $\Gamma \vdash x : A$

($\to$-I) $\dfrac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$

(Ω) $\Gamma \vdash M : \Omega$

($\to$-E) $\dfrac{\Gamma \vdash M : A \to B \;\&\; \Gamma \vdash N : A}{\Gamma \vdash MN : B}$

($\leq$) $\dfrac{\Gamma \vdash M : A \;\&\; A \leq B}{\Gamma \vdash M : B}$

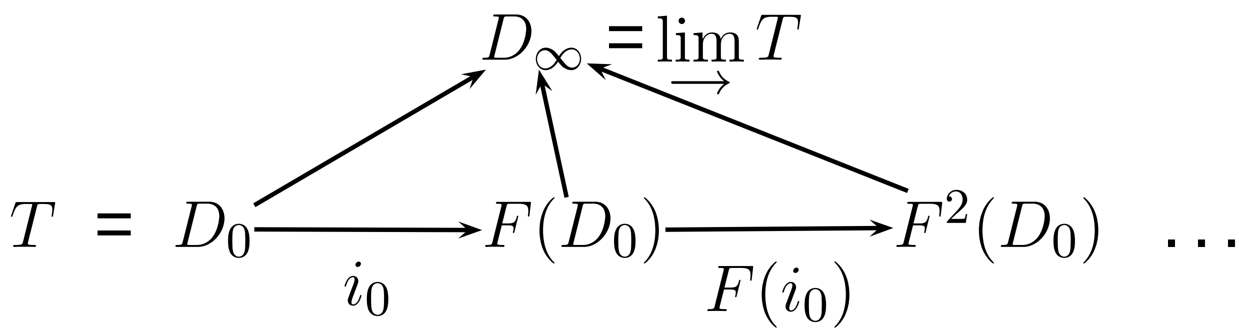
($\cap$-I) $\dfrac{\Gamma \vdash M : A \;\&\; \Gamma \vdash M : B}{\Gamma \vdash M : A \cap B}$

Property:

$$[\![M]\!]_\rho = \{A \mid \Gamma \vdash A \text{ such that } \Gamma \models \rho\}$$

Categorical construction of λ-models...

The paradigmatic λ-model: Scott D_∞

- Take the endofunctor over **ALG**^{ep}: $F(X) = [X \to X]$
- 

  with: $D_0 = \{\bot, \top\}$, $i_0^R(f) = f(\bot)$, $F(i_0) = i_0^R \to i_0$.
- $D_\infty \underset{G}{\overset{F}{\rightleftarrows}} [D_\infty \to D_\infty]$, the COF of F, satisfies the equation $X = [X \to X]$.

. . . and description via filter models

Presentation via Intersection Type Theories of Scott D_∞

- $\Pi^{Sc} : A = \Omega \mid \phi \mid A \cap A \mid A \to A$;
- Σ^{Sc}: the intersection type theory over Π^{Sc} generated by the axioms:

$$\begin{array}{ll} (\Omega\text{-}\eta) & \Omega \sim \Omega \to \Omega \\ (Sc) & \phi \sim \Omega \to \phi \end{array}$$

Theorem

$$\langle \mathcal{F}^{Sc}, F^{Sc}, G^{Sc} \rangle \simeq \langle D_\infty, F, G \rangle$$

Some papers

- [BCD] Barendregt H., Coppo M., Dezani M. *A filter lambda model and the completeness of type assignment system* , J. of Symbolic Logic 48(4), pp. 931–940 (1984)
- [CDHL] Coppo M., Dezani M., Honsell F., Longo G. *Extended type structures and filter lambda models*, Logic Colloquium '82, North-Holland, pp. 241–262 (1984)
- [CDZ] Coppo M., Dezani M., Zacchi M. *Type theories, normal forms, and* D_∞*-lambda-models*, Inf. and Computation, 72(2), pp. 85–116 (1987)
- [HR] Honsell F., Ronche della Rocca S. *An approximation theorem for topological lambda models and the topological incompleteness of lambda calculus* J. Comput. System Sci., 45(1) pp. 49–75 (1992).
- [AO] Abramsky S., Ong L. *Full abstraction in the lazy lambda calculus*, Inform. and Comput. 105(2) pp. 159-267 (1993)
- [HL] Honsell F., Lenisa M *Semantical analysis of perpetual strategies in* λ*-calculus*, TCS 212(2), pp. 182–209 (1999)
- [DGL] Dezani M., Ghilezan S., Likavec S. *Behavioural invers limit models*, TCS 316, pp. 49–74 (2004)
- [ADL] Alessi F., Dezani M., Lusin S. *Intersection types and domains operators*, TCS 316 , pp. 25–47 (2004)
- [DHM] Dezani M., Honsell F, Motohama Y. *Compositional Characterization of* λ*-terms using intersection types*, TCS 340(3), pp. 459–495 (2005)

. . . some results

Filter models are useful for:

- Describing existing models (in **ALG**)
- *Synthesizing "ad hoc" λ-models for proving specific properties of λ-terms*

- [BCD]: first filter model brought to a broad audience. A completeness result proved for set-theoretic semantics of types. Normalizing λ-terms are characterized.

- [CDZ]: The class of normalizing λ-terms is (again) characterized. The class of persistengly normalizing terms is characterized. Approximation theorems are proved.

- [CDHL] anticipation of Abramsky's Domain Theory. The first example of "irregular" filter model is presented. - [HL]: Strongly normalizing λ-terms are characterized.

- [HR]: Special purpose filter model for equating certain fixed point combinators. Proved Incompleteness result for continuous semantics.

- [DHM]: Head normalizing terms are characterized.

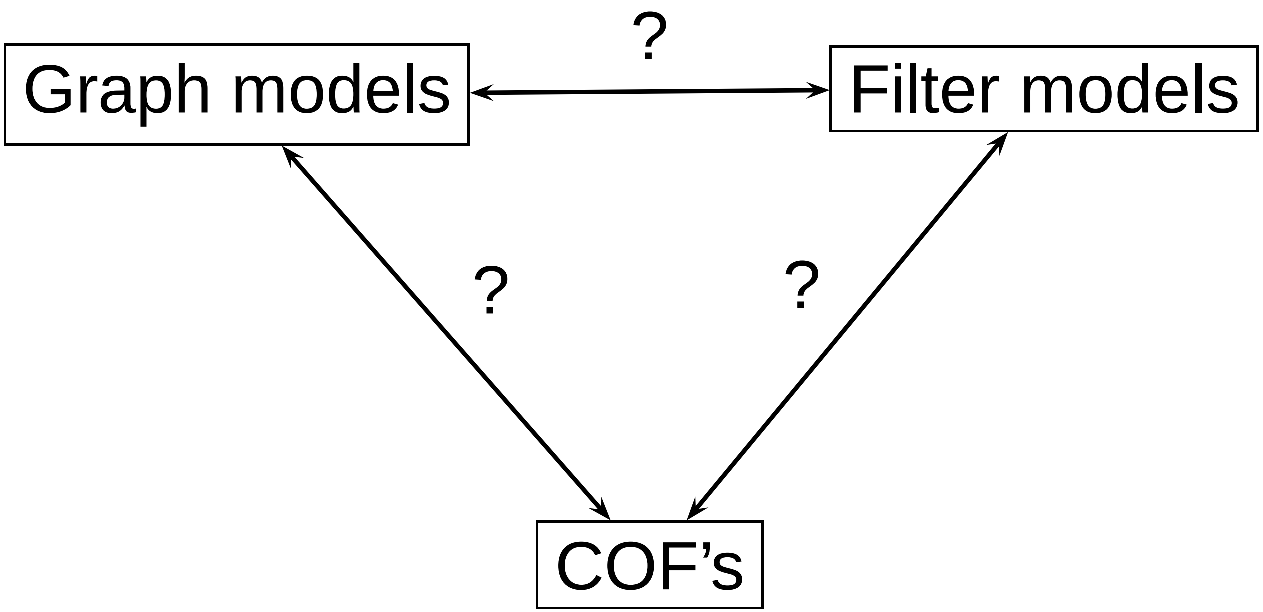
- [DGL]: Nine class of terms are characterized by two filter models.

- [ADL]: Semantic proof of easiness of various lambda terms.

- [DHL]: (Generalized) filter models applied to game semantics.

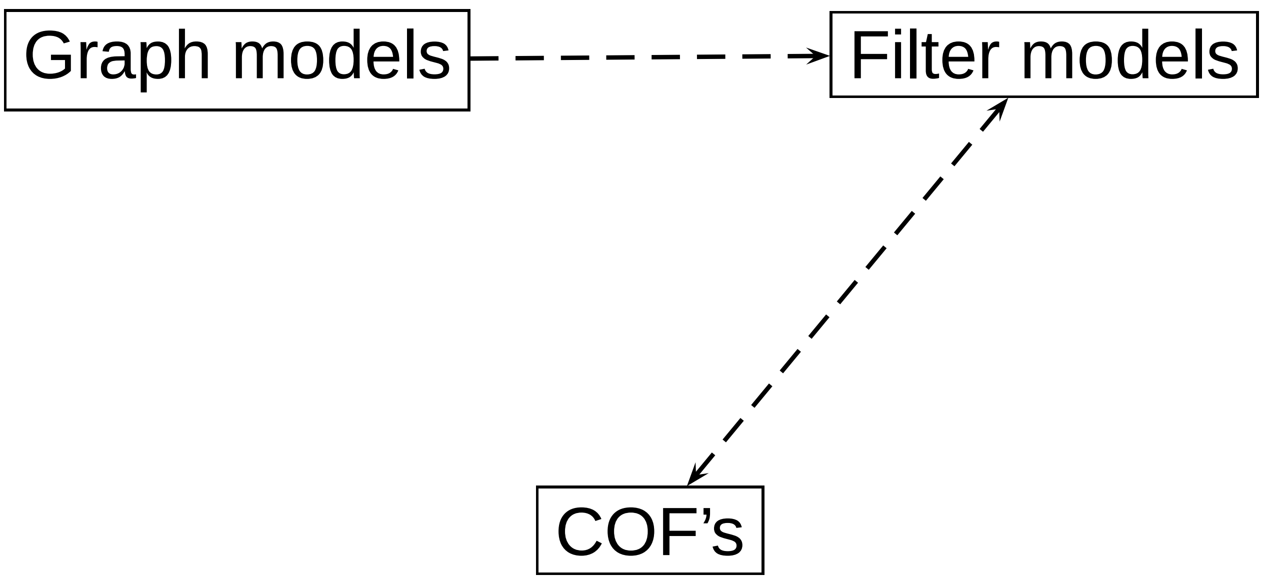
General question

Which connections relate the various kind of models?



General question

Which connections relate the various kind of models?



When dashed arrows work, the correspondences are accounted for by the theory of Scott's Information Systems, or Abramsky's work on Domain Logics, or [CDHL].

The state of the art

- No investigation between graph models and COF’s.
- Countable meet-semilattices with top are Stone dual to ω-algebraic lattices, and filter models fall inside the Abramsky framework (see Abramsky’s *Domain Theory in Logical Form*) as possibile *domain logics* for certain ω-algebraic lattices. As a consequence:
  - Graph models can be presented using filter models (new Barendregt book “Typed Lambda Calculus” - vol. 1, to appear).
  - Classical λ-models which are COF’s (for instance the Scott and Park models) has been usefully presented as filters models (see for instance [HR]).
- According to the choice of functors, certain COF’s cannot be described as filter models.
- Conversely, some filter models are not COF’s. (see [CDHL], [ADL]).
- *Almost all* “ad hoc” filters model in the literature are COF’s.

The purpose of the paper

- To find a restricted class of filter models and (solutions of) domain equations which correspond each other.
- The new framework *must encompass filter models of the literature*.

Two motivations for this plan:

- Describing a filter model as a COF clarifies the role of the syntactic choices on which a filter model is built (discriminating what is essential from what is not).
- - The COF's that are described via filter models, may take advantage of finitary techniques (e.g. TAS).
  - The presentation of COF's via filter models is more intuitive, since it does not use the category theory apparatus, but just set theoretic constructions.

Plan of the paper

- The starting point: The most relevant filter models in the literature *ARE ALWAYS COF's* (but for one case: [CDHL] filter model).
- The plan of the paper:
  - Extract the (implicit) syntactic restrictions present in the filter models of the literature.
  - These restrictions lead to the notion of *disciplined* ITT and *disciplined* filter structure.
  - Study the tight correpondence between domain equations and disciplined filter models.

Selection of ITT's

- Equated ITT's: Let $\alpha, \alpha' \not\sim \Omega$ atoms:
  - $\boxed{\exists \beta_i, \gamma_i.\ \alpha \sim \bigcap_{i\in I}(\beta_i \to \gamma_i)}$ (I finite)
  - If $\alpha \leq \alpha'$, $\alpha \sim \bigcap_{i\in I}(\beta_i \to \gamma_i)$ $\alpha' \sim \bigcap_{i\in J}(\beta'_j \to \gamma'_j)$, then

    $$\boxed{\forall i' \in I'. \bigcap\{\gamma_i \mid \beta'_j \leq \beta_i\} \leq \gamma'_j.}$$

- Split ITT's: If $\alpha \neq \Omega$ and $\exists i.\ \Omega \not\leq B_i$,

  $$\boxed{\alpha \not\lesseqgtr \bigcap_{i\in I}(A_i \to B_i)}$$

Moreover, based on derivability of (†) $\boxed{\Omega \sim \Omega \to \Omega}$, define

- Natural ITT's: (†) is derivable.
- Lazy ITT's: (†) is not derivable.

Proposal: work with *disciplined* ITT's

Definition An ITT Σ^{∇} is disciplined if it combines a property $P \in \{\textit{equated}, \textit{split}\}$ with a property $Q \in \{\textit{natural}, \textit{lazy}\}$.
So four cases of disciplined ITT's are possibile:

- natural equated;
- lazy equated;
- natural split;
- lazy split.

$\mathcal{F}^{\nabla}$ is *disciplined* if so is Σ^{∇}.

Theorem: A disciplined filter structure Σ^{∇} is *always* a λ-model, since it is reflexive.

Features of disciplined filter models

- The restriction to disciplined filter models makes sense since the filter models presented in the literature are so (but for [CDHL] filter model).
- The semantic counterpart of disciplined filter models is clear.
- Previous scattered proofs of isomorphisms between COF's and filter models can be viewed as particular cases of a more general proof based on properties of disciplined ITT's.

Non-disciplined filter models

- From the literature: the [CDHL] filter model (built as an example of non-reflexive filter model).
- Filter models based on *inequality* axioms, for instance:
  - $\mathbb{T}^\flat : A = \Omega \mid \phi \mid A \cap A \mid A \to A$;
  - $\Sigma^\flat$: the intersection type theory over $\mathbb{T}^\flat$ generated by the axioms:

$$\begin{array}{ll} (\Omega\text{-}\eta) & \Omega \sim \Omega \to \Omega \\ (\flat) & \phi \leq \phi \to \phi \end{array}$$

$\mathcal{F}^\flat$ cannot be framed as a colimit in **ALG**.

The classification result of the paper

Domain equations for disciplined filter models

[BCD]	Nat. split	$F(X) = \mathcal{B} \times [X \to X]$
none	Lazy split	$F(X) = \mathcal{B} \times [X \to X]_\perp$
Scott, Park, [HR], [ADL] [CDZ], [DHM], [DGL]	Nat. eq'd	$F(X) = [X \to X]$
[AO]	Lazy eq'd	$F(X) = [X \to X]_\perp$

Theorem 1 Any colimit which solves one of the four domain equations above can be defined in a canonical way as a disciplined filter model.

The classification result of the paper

Domain equations for disciplined filter models

[BCD]	Nat. split	$F(X) = \mathcal{B} \times [X \to X]$
none	Lazy split	$F(X) = \mathcal{B} \times [X \to X]_\perp$
Scott, Park, [HR],[ADL] [CDZ], [DHM], [DGL]	Nat. eq'd	$F(X) = [X \to X]$
[AO]	Lazy eq'd	$F(X) = [X \to X]_\perp$

Theorem 2 For each disciplined filters model $\mathcal{F}^\nabla$ (in particular: any of the literature) there is a canonical triple F, D_0, $i_0 : D_0 \to F(D_0)$ such that

$$\mathcal{F}^\nabla \simeq \varinjlim \langle F^n(D_0), F^n(i_0) \rangle$$

Therefore any disciplined $\mathcal{F}^\nabla$ is a COF.

Open problems

- To investigate on the relation between graph models and COF's;
- Which is the categorical framework for non-disciplined filter models? Can they be viewed as colimits of functors in some more general category of domains?
- Concerning the relation syntax/semantics, which are the semantic consequences of non-standard choice of axioms for defining ITT's/filter models?

[CDHL] filter model

- $\Pi^{cdhl} : A = \Omega \mid \phi_n \mid A \cap A \mid A \to A \quad (n \in \omega)$
- Σ^{cdhl}: the intersection type theory over Π^{cdhl} generated by the axioms:

$$
\begin{array}{ll}
(\Omega\text{-}\eta) & \Omega \sim \Omega \to \Omega \\
(cdhl) & A \leq A[\phi/B]
\end{array}
$$

Fact: $\mathcal{F}^{cdhl}$ is a lambda model but it is not reflexive.