Certification of proving termination of term rewriting by matrix interpretations

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Outline

1. Background
   - Termination of Term Rewriting
     - Term Rewriting
     - Termination of Term Rewriting
     - Automation of Proving Termination
   - Certification of Termination
     - CoLoR project: Certification of Termination Proofs
     - Certified Competition
2. Formalization of Matrix Interpretations
   - Matrix Interpretations Method
   - Monotone algebras
   - Matrices
   - Matrix interpretations
3. Conclusions & Future Work

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- Term rewriting is a model of computations.

Example

$$0 + y \to y$$
$$s(x) + y \to s(x + y)$$
$$0 * y \to 0$$
$$s(x) * y \to (x * y) + y$$
$$fact(0) \to s(0)$$
$$fact(s(x)) \to s(x) * fact(x)$$

Term Rewriting

- Term rewriting is a model of computations.

Example

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$fact(s(s(s(0)))) \to^{+} 6$

- One of the most important properties of term rewriting is termination.

Definition

A term rewriting system (TRS) is terminating if it does not admit infinite reductions.

- In general the problem is undecidable.
- However, there is a (ever increasing) number of techniques for proving termination of term rewriting.

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$$aa \to aba$$

Automation of Proving Termination

- Recently the emphasis is on automation.
- There is a number of tools for proving termination automatically. (AProVE, Cariboo, Cime, JamBox, MatchBox, MultumNonMulta, MuTerm, Teparla, Torpa, TPA, TTT, TTTbox, . . .)
- An annual termination competition is organized where those tools compete on a number of problems.
- Both the tools and proofs produced by them are getting more and more complex.
- Reliability of such tools is a challenge and indeed every year we observe some disqualifications due to erroneous proofs.

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CoLoR overview

CoLoR

CoLoR: Coq Library on Rewriting and Termination.
Goal: certification of termination proofs produced by various termination provers.
Project started in March 2004 by Frédéric Blanqui.

How to do that? CoLoR approach:

- TPG: common format for termination proofs.
- Tools output proofs in TPG format.
- CoLoR: a Coq library of results on termination.
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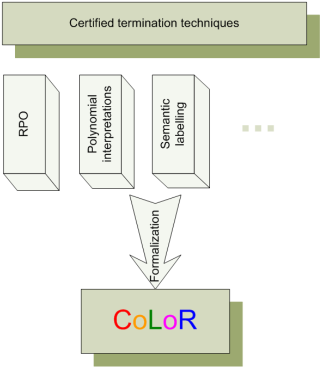

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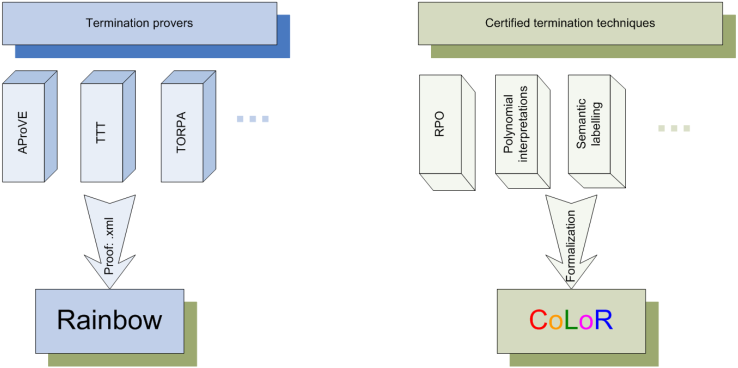

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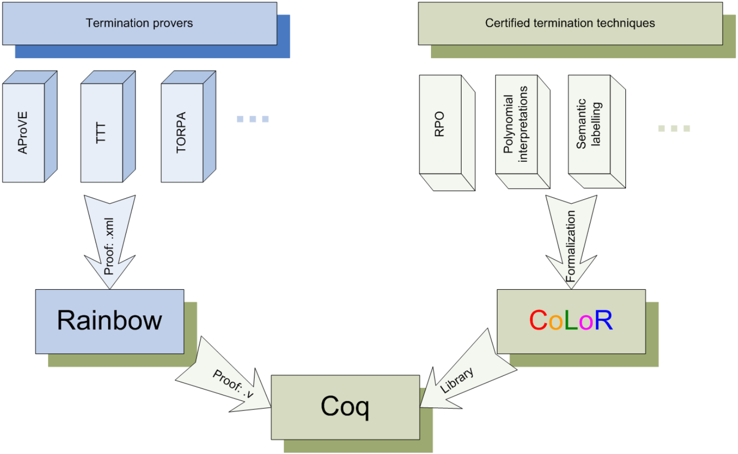

Certified competition

- In the termination competition in 2007 a new "certified" category was introduced.
- Participants:
  - CiME+ A3PAT
  - TPA+ CoLoR
  - $\mathrm{T_TT_2}$ + CoLoR
- TPA+ CoLoR was the winner with the score of 354.
- Every successful proof of TPA was using matrix interpretations.

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General idea

- A popular approach is interpretation into a well-founded monotone algebra.
- Domain: $\mathbb{N}$, $f(x_1, \ldots, x_n)$ interpreted as polynomial $\mathbb{N}[x_1, \ldots, x_n]$
  $\Longrightarrow$ polynomial interpretations (Lankford '79)
- Domain: $\mathbb{N}^d$, $f(\vec{x_1}, \ldots, \vec{x_n}) = A_1\vec{x_1} + \ldots + A_n\vec{x_n} + \vec{b}$, with
  $A_i \in \mathbb{N}^{d \times d}$, $\vec{b} \in \mathbb{N}^d$
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Example

Example

$$a(a(x)) \to a(b(a(x), c))$$

$$a(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$b(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} y$$
$$c = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$[b(a(x), c)] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x$$
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$$\begin{pmatrix} u_1 \\ \cdots \\ u_d \end{pmatrix} \gtrsim \begin{pmatrix} v_1 \\ \cdots \\ v_d \end{pmatrix} \text{ iff } \forall i, u_i \geq_{\mathbb{N}} v_i$$
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Monotone algebras

Definition (An extended weakly monotone Σ-algebra)

An *extended weakly monotone Σ-algebra* $(A, [\cdot], >, \gtrsim)$ is a Σ-algebra $(A, [\cdot])$ equipped with two binary relations $>$, $\gtrsim$ on A such that:

- $>$ is well-founded;
- $> \cdot \gtrsim \subseteq >$;
- for every $f \in \Sigma$ the operation $[f]$ is monotone with respect to $>$.

Theorem

Let $\mathcal{R}, \mathcal{R}'$ be TRSs over a signature Σ, $(A, [\cdot], >, \gtrsim)$ be an extended monotone Σ-algebra such that:

- $[\ell, \alpha] \gtrsim [r, \alpha]$ *for every rule* $\ell \to r$ *in* $\mathcal{R}$, *for all* $\alpha : \mathcal{X} \to A$ *and*
- $[\ell, \alpha] > [r, \alpha]$ *for every rule* $\ell \to r$ *in* $\mathcal{R}'$ *and for all* $\alpha : \mathcal{X} \to A$.

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Formalization of monotone algebras

- Monotone algebras are formalized as a functor.
- We additionally require $>_{\mathcal{T}}$ and $\gtrsim_{\mathcal{T}}$ to be decidable.
(where $s >_{\mathcal{T}} t \equiv \forall \alpha : \mathcal{X} \to A, [s, \alpha] > [t, \alpha]$)
- More precisely the requirement is to provide a relation $\gg$, such that
  - $\gg \subseteq >_{\mathcal{T}}$ and
  - $\gg$ is decidable
  - similarly for $\gtrsim$.
- The structure returned by the functor contains all the machinery required to prove (relative)-(top)-termination in Coq.

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Formalization of matrices

- Matrices over arbitrary semi-ring of coefficients.
- a number of basic operations over matrices such as:

$$[\cdot], \quad M_{i,j}, \quad M + N, \quad M * N, \quad M^T, \ \ldots$$

- and a number of basic properties such as:
  - $M + N = N + M$,
  - $M * (N * P) = (M * N) * P$
  - monotonicity of $*$
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Polynomial interpretations in the setting of monotone algebras

- $A = \mathbb{Z}$,
- $> \; = \; >_{\mathbb{Z}}$, $\gtrsim = \geq_{\mathbb{Z}}$,
- interpretations represented by polynomials $[f(x_1, \ldots, x_n)] = P_{\mathbb{Z}}(x_1, \ldots, x_n)$,
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- fix a dimension d,
- $A = \mathbb{N}^d$,
- $(u_1, \ldots, u_d) \gtrsim (v_1, \ldots, v_d)$ iff $\forall i, u_i \geq_{\mathbb{N}} v_i$,
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- interpretations represented as:
  $[f(x_1, \ldots, x_n)] = M_1 x_1 + \ldots + M_n x_n + v$
  where $M_i \in \mathbb{N}^{d \times d}$, $v \in \mathbb{N}^d$,
- $>_{\mathcal{T}}$ and $\gtrsim_{\mathcal{T}}$ are decidable in this case but thanks to introducing $\gg$ we do not need to prove completeness of their characterization.
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Conclusions & Future Work

We presented:

- formalization of the matrix interpretations method,
- that allowed TPA+ CoLoR to win the certified competition in 2007.

Future work:

- extension to arctic matrices (max/plus semi-ring over $\mathbb{N} \cup \{-\infty\}$).
- Formalization of further termination techniques.
- Collaboration with termination tools' authors to extend applicability of CoLoR.

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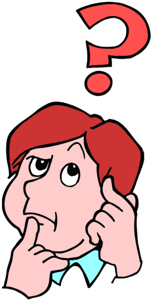
The end

http://color.loria.fr



Thank you for your attention.

Homework

If you are bored in the evening (or like puzzles) are the following systems terminating:

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$$aa \longrightarrow bc$$
$$bb \longrightarrow ac$$
$$cc \longrightarrow ab$$

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