Reachability Games of Ordinal Length

Julien Cristau Florian Horn

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Plan

Motivations

Infinite games

Ordinals

Solving reachability games

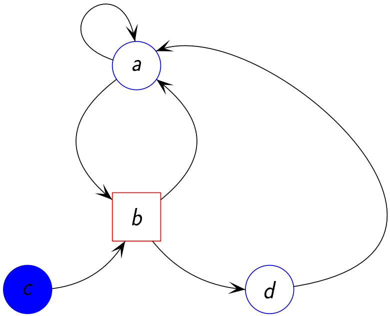
Conclusion

Motivations

- Verification of open systems, controller synthesis
  - Games are very useful
  - One player (Eve) corresponds to the system, the opponent (Adam) represents the system
- Modelisation of systems where an unbounded number of events happen in finite time
  - timed systems, real time models
  - so-called Zeno behaviours

Games

- Finite graph $G = (V, E)$
- Partition $V = V_E \cup V_A$
- 2 players, Eve and Adam; Eve plays in V_E and Adam in V_E
- Winning condition



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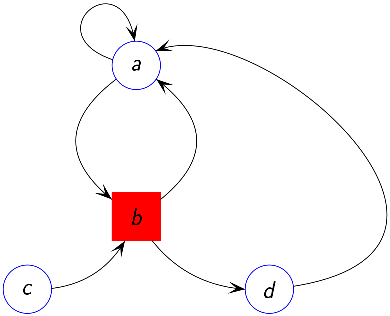

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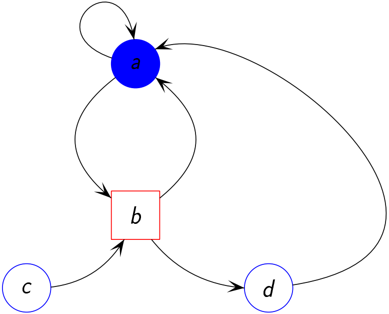

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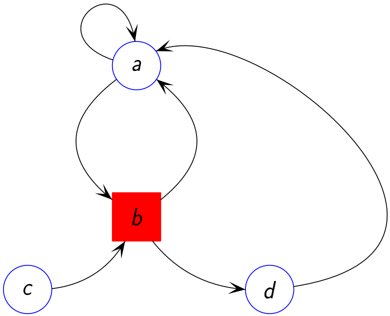

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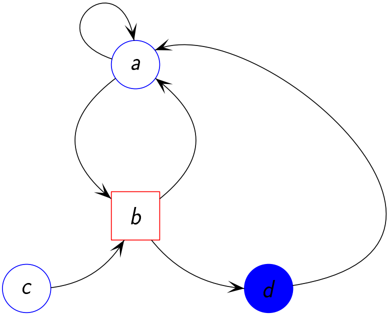

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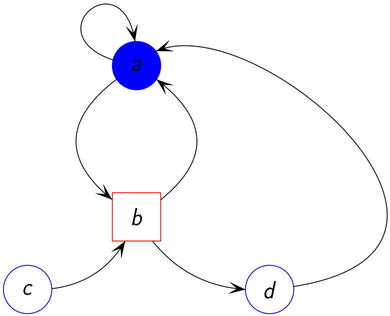
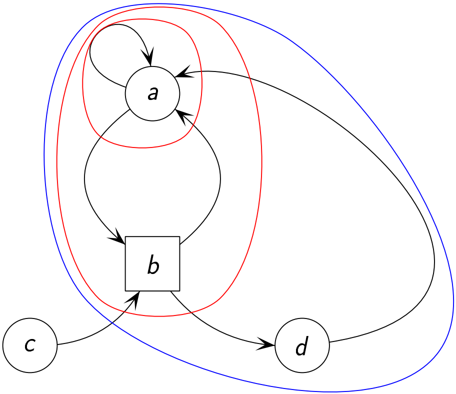


Muller games

Winning condition: Eve wins if the set of states visited infinitely often is in $\mathcal{F}$.

Example: Eve wins $\{a, b, d\}$, Adam wins $\{a, b\}$ and $\{a\}$.



A play is an infinite word, like *cbabdababdabdabababab da* ....

Problems

A game is given by a partitioned graph and a winning condition.

We want to know:

- whether the game is determined (one of the players has a winning strategy)
- given an initial state, which is the winning player
- how to compute a winning strategy

Theorem

Muller games are determined (Martin).
Finding the winner is PSPACE*-complete (Hunter and Dawar).*

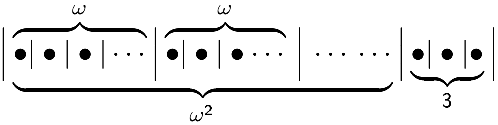
Beyond ω

We want models of systems where infinitely many actions can happen in finite time (Zeno behaviours).

A play is now a word of ordinal length, such as $((ab)^{\omega} c)^{\omega} (ba)^{\omega} d$

Examples:

- ω
- $\omega^2 + 3$



Games

We add limit transitions to the arena.

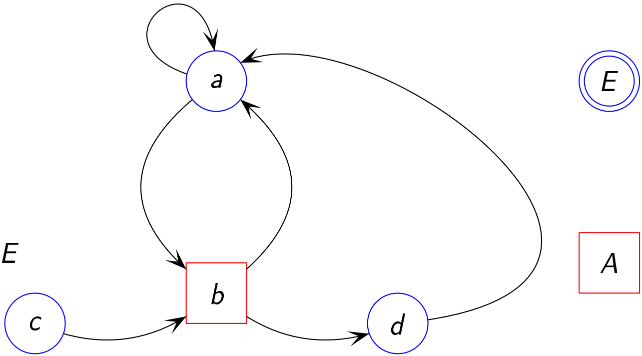
$\{a\} \longrightarrow c$

$\{a, b\} \longrightarrow d$

$\{a, b, c\} \longrightarrow A$

$\{a, b, d\} \longrightarrow A$

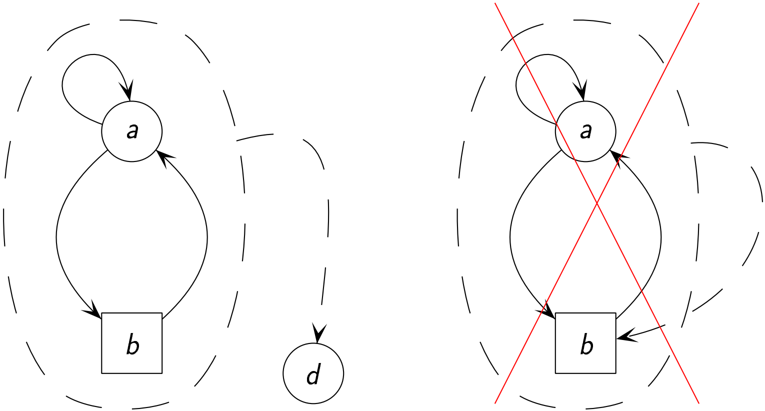
$\{a, b, c, d\} \longrightarrow E$



Winning condition: Eve wins when the token reaches vertex E.

A technical restriction

Limit transitions of the form $P \to q$ where $q \in P$ are forbidden.



With this condition, plays can't be longer than ω^ω.

Problems

Reachability game of ordinal length

- a graph with limit transitions,
- two players,
- a state to reach.

The questions are:

- is the game determined?
- if yes, which player has a winning strategy?
- can his strategy be computed?

Notice that the length of a play is not fixed.
The game stops when one of the players wins.

Results

Theorem

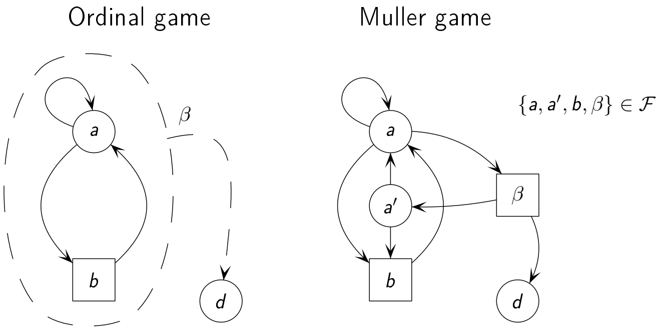
Reachability games of ordinal length $< \omega^\omega$ are determined.

Theorem

Finding the winner is PSPACE*-complete.*

Idea of the proof:
the game is reduced to a Muller game, where we can determine the winner.

Reduction



A winning strategy in the Muller game corresponds to a winning strategy in the ordinal game.

Conclusion

Results:

- One of the players always wins (determinacy)
- Finding the winner with same complexity as for traditional Muller games (PSPACE-complete)

Remaining questions:

- How much memory is needed?
- Are there classes where it is finite?
- Can we lift the restriction to ordinals $< \omega^\omega$?