On the undecidability of the tiling problem

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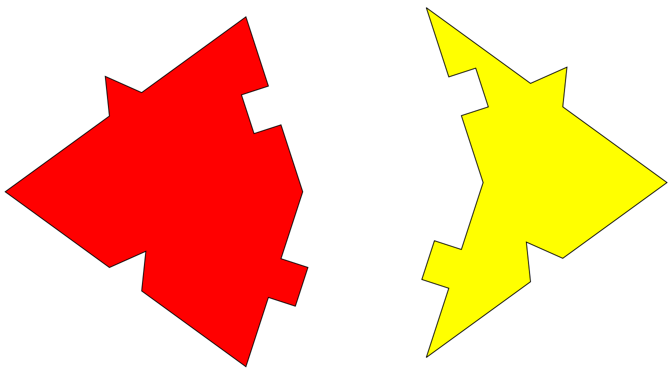
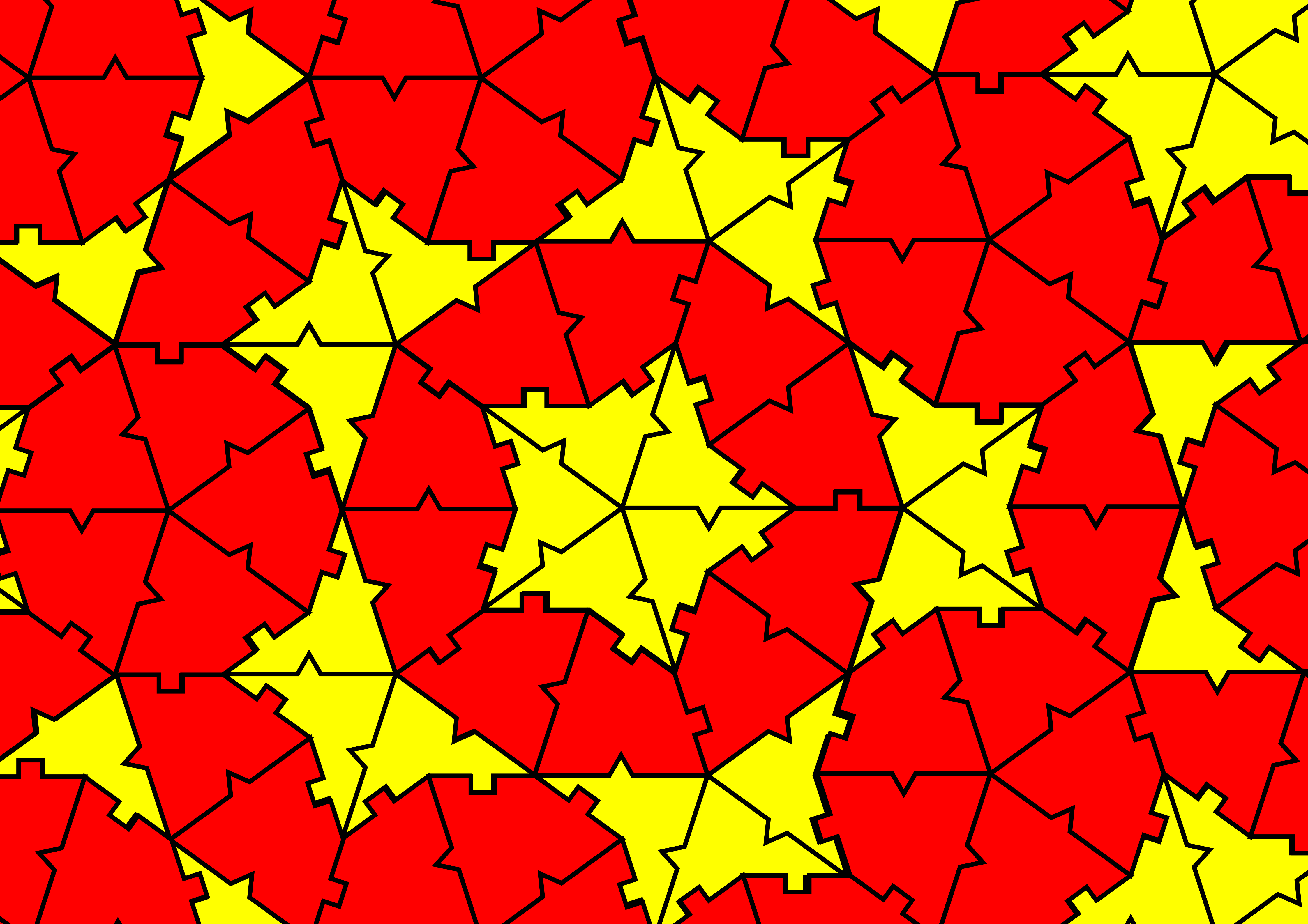
Consider the following decision problem, the **tiling problem**:
Given a finite set of tiles (say, for example, polygons), is it possible to tile the infinite plane with copies of the tiles ?

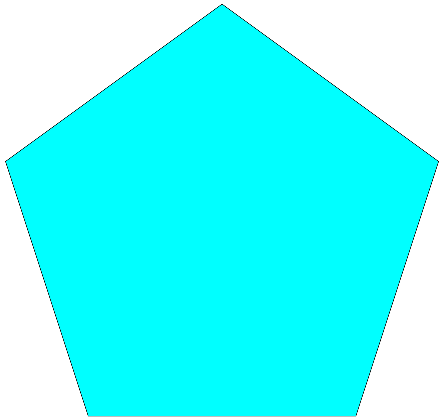
For instance, can one tile the plane with copies of



By a tiling we mean a covering of the plane without overlaps (i.e. two tiles may only overlap in their boundary).



But there are other tile sets that do not admit any tiling, e.g. the regular pentagon



does not tile the plane.

So is there an algorithm to tell which tile sets admit a plane tiling ?

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R.Berger 1966: No, the tiling problem is undecidable.

R.Berger: Undecidability of the Domino Problem. *Memoirs of the American Mathematical Society* **66, 72 pp., 1966.**

A simplified version (but based on Berger's ideas) was given by R.M.Robinson in 1971.

R.M.Robinson. Undecidability and nonperiodicity for tilings of the plane. *Inventiones Mathematicae* **12, 177–209, 1971.**

In this talk we present a new, quite different proof.

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• The new proof is simpler. It is based on simple algebra, it is precise and easy to verify.

• The same technique works in other set-ups as well. In particular, the same approach shows that the tiling problem on the **hyperbolic plane** is undecidable. This problem was posed in Robinson's 1971 paper, and investigated later in more depth in

R.M.Robinson: Undecidable tiling problems in the hyperbolic plane. *Inventiones Mathematicae* **44, 259–264, 1978.**

The hyperbolic version was proved undecidable in 2007 independently (and with a very different proof) by M.Margenstern.

Outline of the talk

- Introduction. Historical perspective.
- Wang tiles
- Aperiodicity. An aperiodic tile set of 14 Wang tiles.
- The immortality problem of piecewise affine transformations.
- Reductions:
  (a) Immortality problem of Turing machines $\longrightarrow$ immortality problem of piecewise affine transformations of $\mathbb{R}^2$.
  (b) Immortality problem of piecewise affine maps $\longrightarrow$ the tiling problem
- Undecidability of the tiling problem in the hyperbolic plane.

Wang tiles

A Wang tile is a unit square tile with colored edges. A tile set T is a finite collection of such tiles. A valid tiling is an assignment

$$\mathbb{Z}^2 \longrightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.

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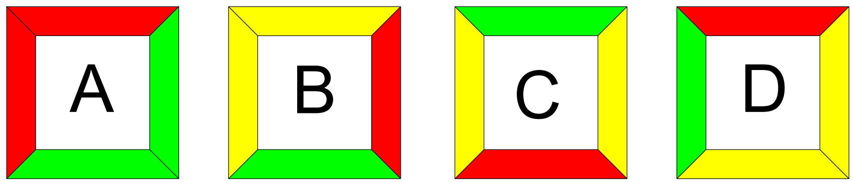
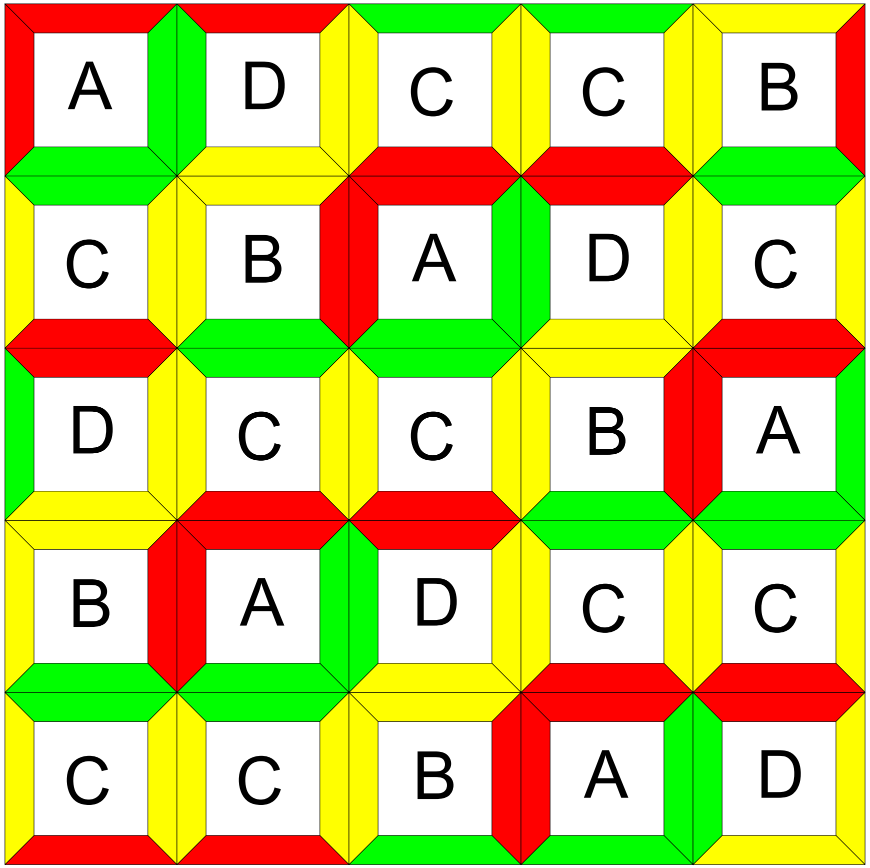
For example, consider Wang tiles



With copies of the given four tiles we can properly tile a 5×5 square...



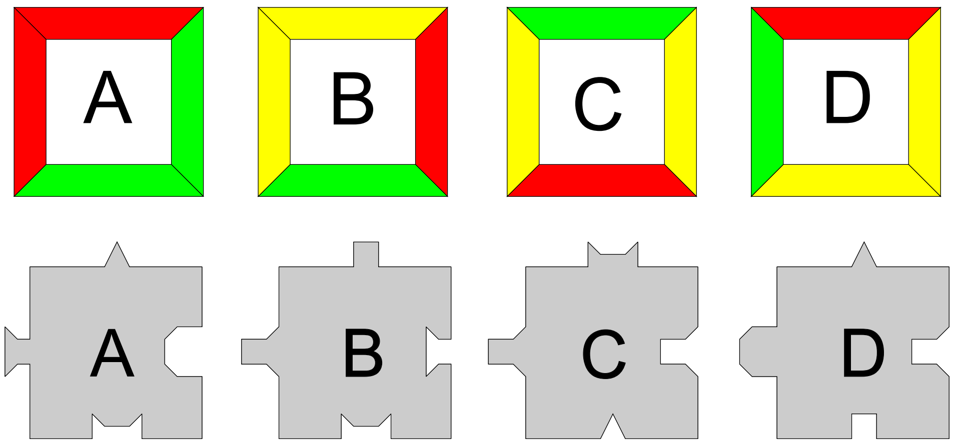
... and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.

The **tiling problem** of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

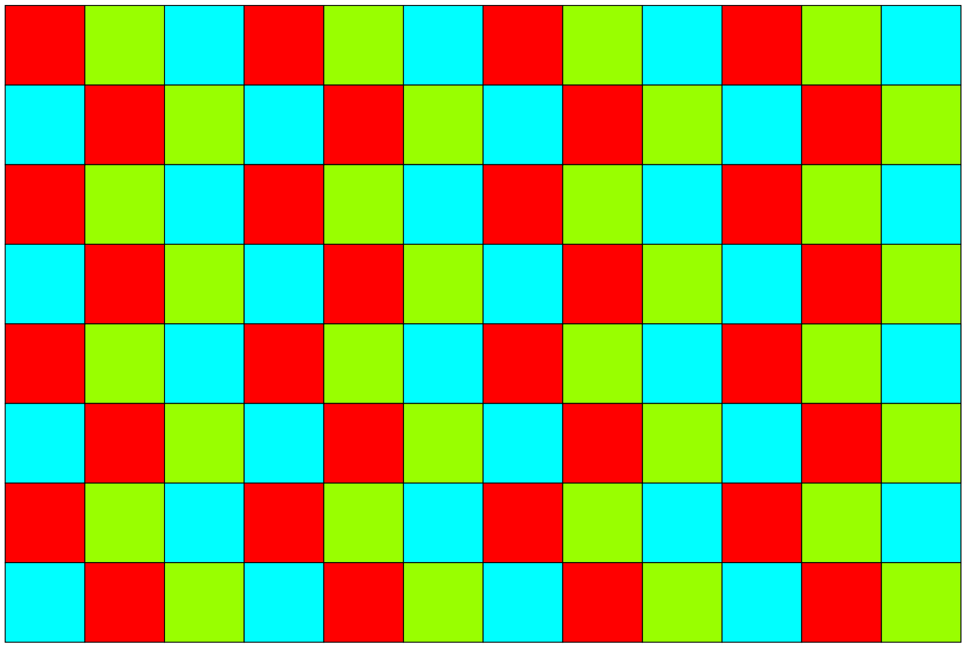
Note: Wang tiles are abstract tiles, but one can effective transform them into equivalent concrete shapes (e.g. polygons with rational coordinates).

For example, we can make each Wang tile into a unit square tile whose left and upper edges have a bump and the right and lower edge has a dent. The shape of the bump/dent depends on the color of the edge. Each color has a unique shape associated with it (and different shapes are used for horizontal and vertical colors).



Aperiodicity

A tiling is called **periodic** if it is invariant under some non/zero translation of the plane. A simple reasoning shows that any Wang tile set that admits a periodic tiling also admits a tiling with a horizontal and a vertical period: the tiling is formed by repeating a rectangular pattern.



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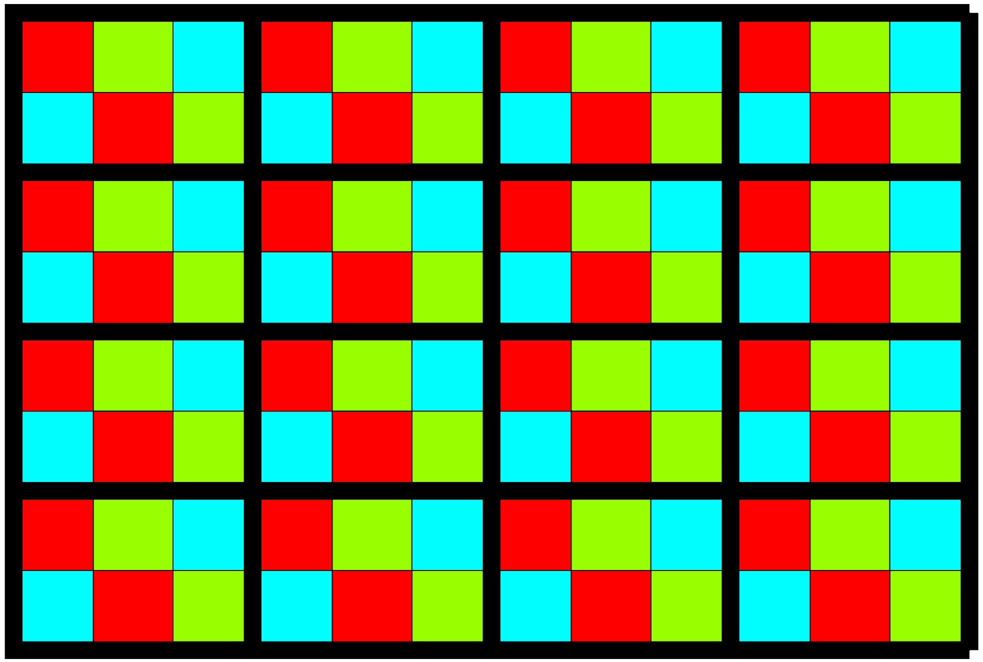

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Berger's original aperiodic tile set contained over 20,000 tiles. Smaller aperiodic sets were soon discovered by various people. The current record is a 13 tile aperiodic set of Wang tiles, due to K.Culik.

If Wang's conjecture had been true and aperiodic tile sets would not exist then the tiling problem would be decidable: One could try all possible tilings of larger and larger rectangles until either

(a) **a rectangle is found that can not be tiled (so no tiling of the plane exists)**, or

(b) a tiling of a rectangle is found such that the colors at left and right sides match and the colors of the top and bottom sides match each other (so a periodic tiling exists).

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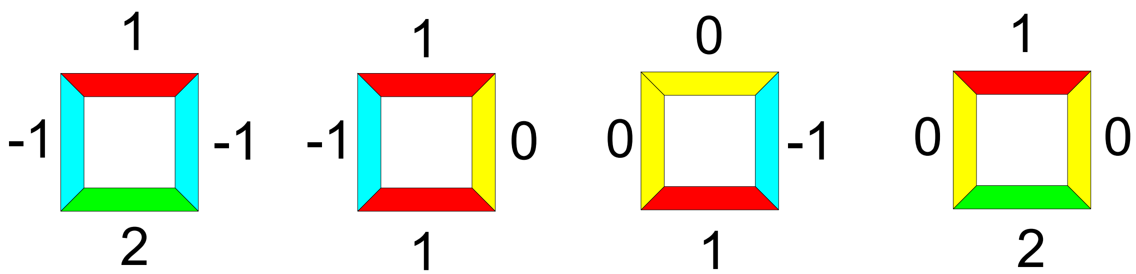
We see that any undecidability proof of the tiling problem must contain (explicitly or implicitly) a construction of an aperiodic tile set.

14 tile aperiodic set

The colors in our Wang tiles are real numbers, for example



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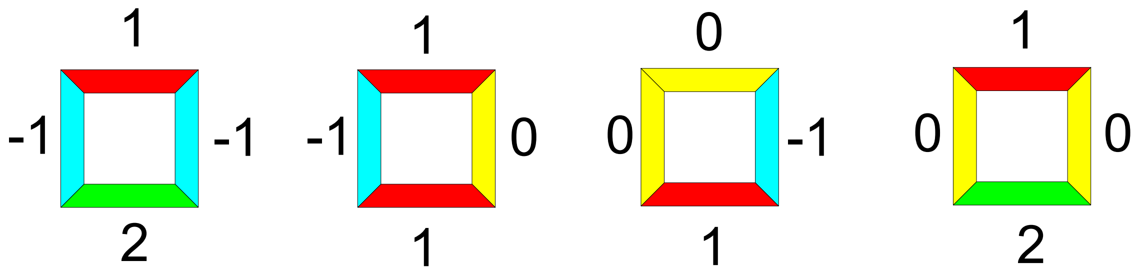
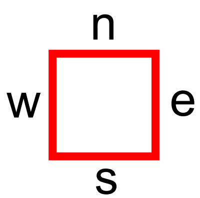


We say that tile



multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

(The "input" n comes from the north, and the "carry in" w from the west is added to the product qn. The result is split between the "output" s to the south and the "carry out" e to the east.)

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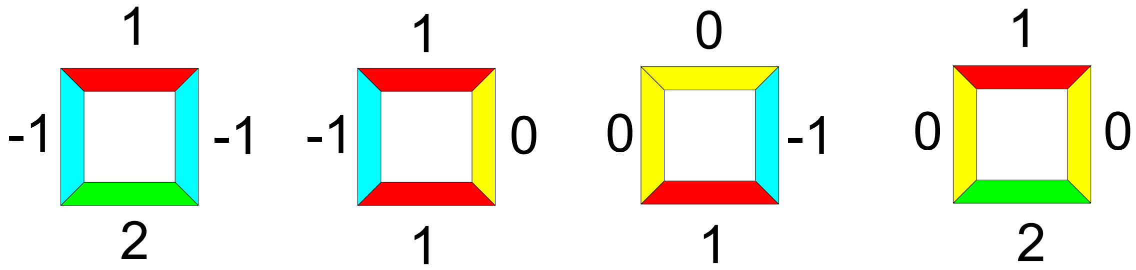

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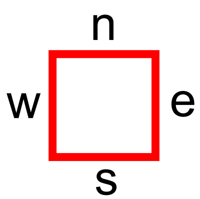
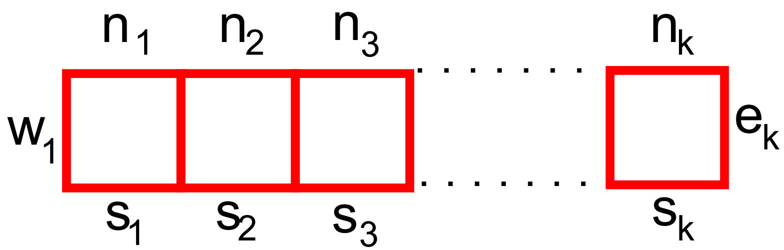


multiplies by number $q \in \mathbb{R}$ if

$$qn + w = s + e.$$

The four sample tiles above all multiply by $q = 2$.

Suppose we have a correctly tiled horizontal segment where all tiles multiply by the same q.



It easily follows that

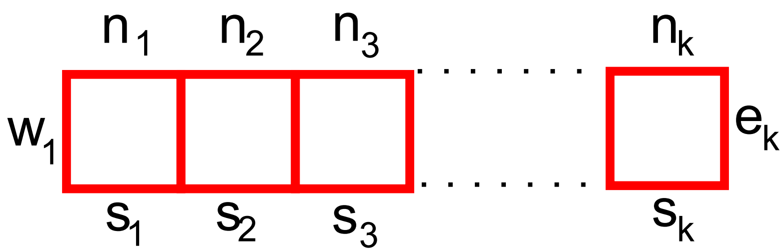
$$q(n_1 + n_2 + \ldots + n_k) + w_1 = (s_1 + s_2 + \ldots + s_k) + e_k.$$

To see this, simply sum up the equations

$$\begin{aligned}
qn_1 + w_1 &= s_1 + e_1 \\
qn_2 + w_2 &= s_2 + e_2 \\
&\vdots \\
qn_k + w_k &= s_k + e_k,
\end{aligned}$$

taking into account that always $e_i = w_{i+1}$.

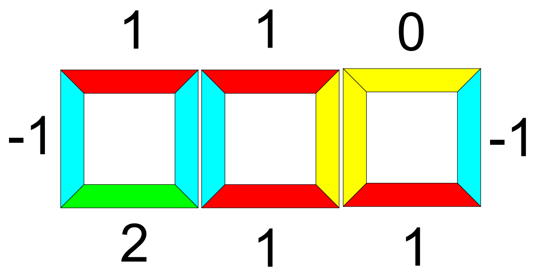
Suppose we have a correctly tiled horizontal segment where all tiles multiply by the same q.



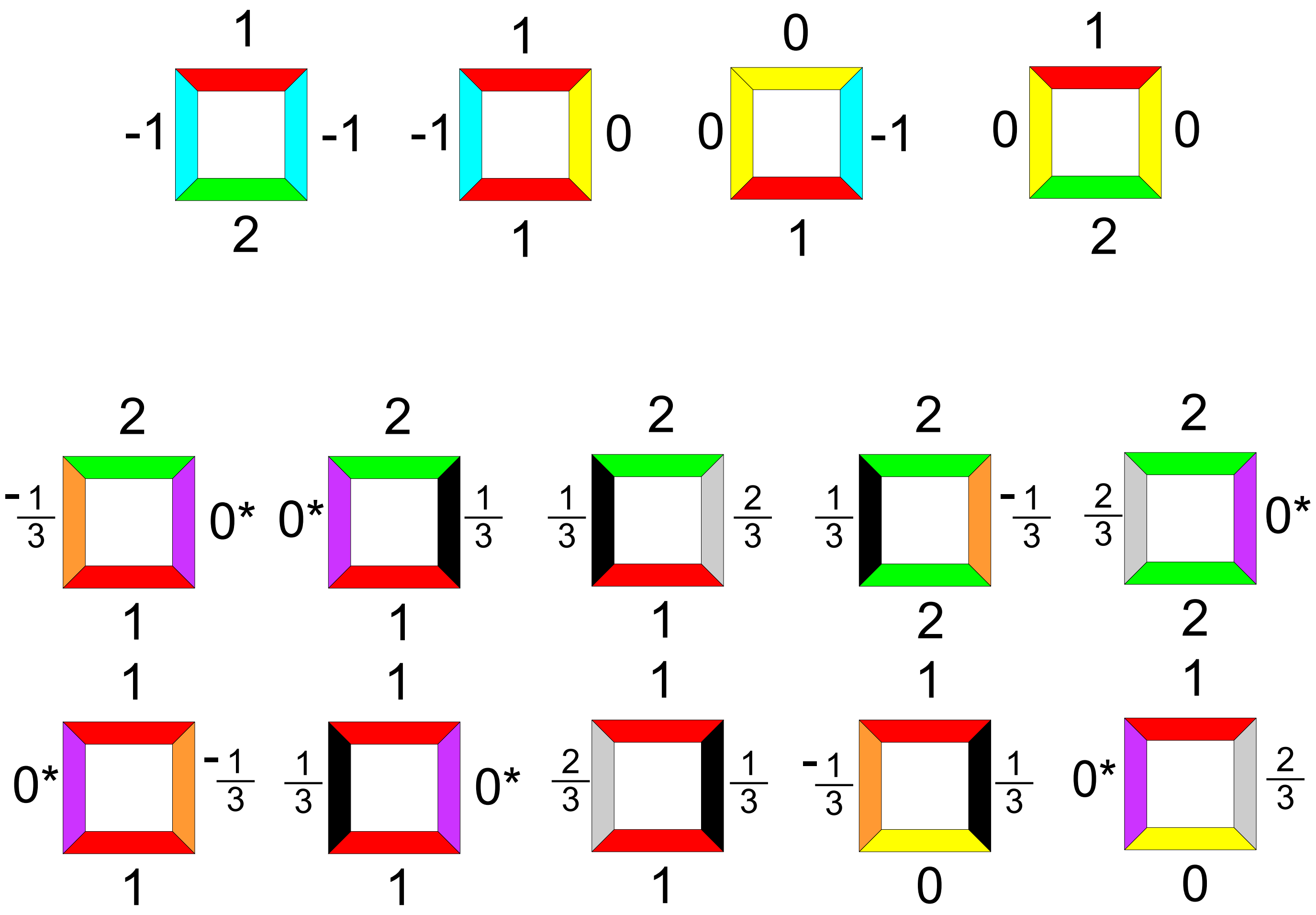
If, moreover, the segment begins and ends in the same color $(w_1 = e_k)$ then

$$q(n_1 + n_2 + \ldots + n_k) = (s_1 + s_2 + \ldots + s_k).$$

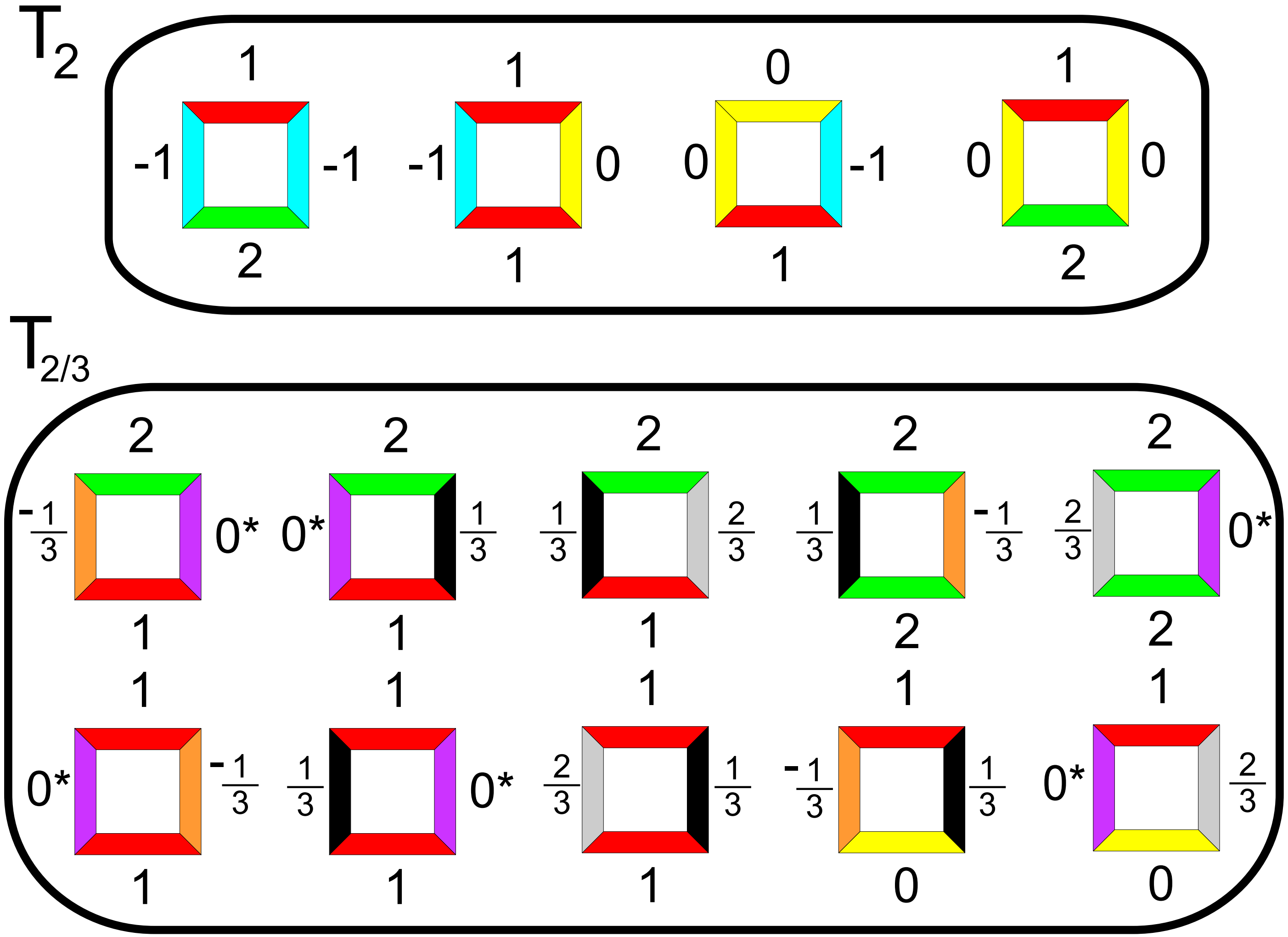
For example, using our three sample tiles that multiply by $q = 2$ we can form the segment



in which the sum of the bottom labels is twice the sum of the top labels.

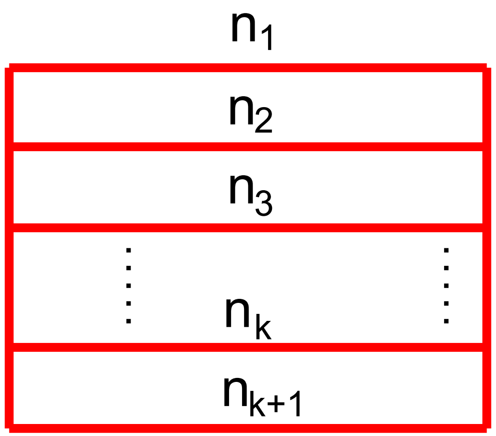


Our aperiodic tile set consists of the four tiles that multiply by 2, together with another family of 10 tiles that all multiply by $\frac{2}{3}$.



Let us call these two tile sets T_2 and $T_{2/3}$. Vertical edge colors of the two parts are made disjoint, so any properly tiled horizontal row comes entirely from one of the two sets.

Let us prove that no periodic tiling exists. Suppose the contrary: A rectangle can be tiled whose top and bottom rows match and left and right sides match.



Denote by n_i the sum of the numbers on the i'th horizontal row (counted from top to bottom). Let the tiles of the i'th row multiply by $q_i \in \{2, \frac{2}{3}\}$.

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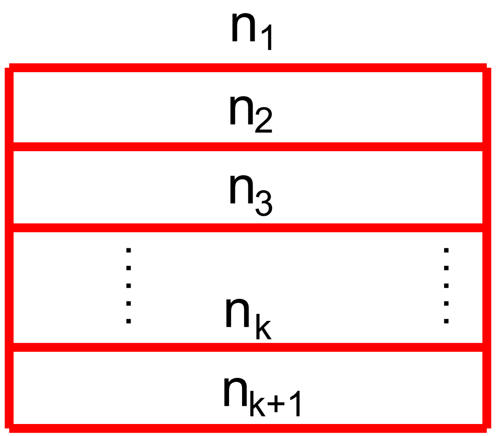

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From our previous discussion we know that $n_{i+1} = q_i n_i$, for all i.

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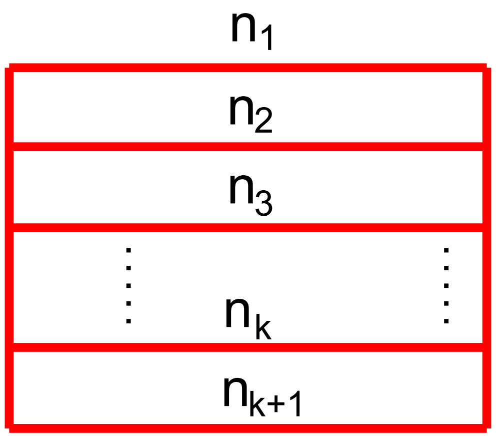

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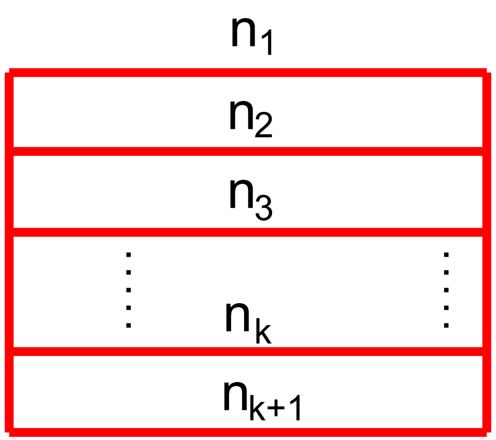

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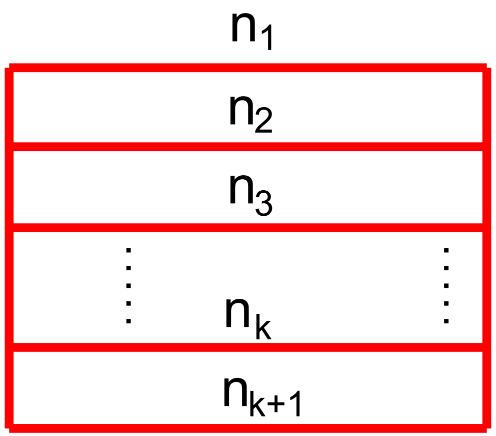

So we have $q_1 q_2 q_3 \ldots q_k n_1 = n_{k+1} = n_1$.

Clearly $n_1 > 0$, so we have $q_1 q_2 q_3 \ldots q_k = 1$. But this is not possible since 2 and 3 are relatively prime: No product of numbers 3 and $\frac{2}{3}$ can equal 1.

Next step: We still need to show that a valid tiling of the plane exists using our tiles. For this purpose we introduce **sturmian** or **balanced** representations of real numbers as bi-infinite sequences of two closest integers.

The representation of any $\alpha \in \mathbb{R}$ is the sequence $B(\alpha)$ whose k'th element is

$$B_k(\alpha) = \lfloor k\alpha \rfloor - \lfloor (k-1)\alpha \rfloor.$$

For example,

$$B(\tfrac{1}{3}) \quad = \quad \ldots 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots$$

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$$\begin{aligned} B(\tfrac{1}{3}) &= \ldots 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots \\ B(\tfrac{7}{5}) &= \ldots 1\ 1\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 2\ 1\ 1\ \ldots \end{aligned}$$

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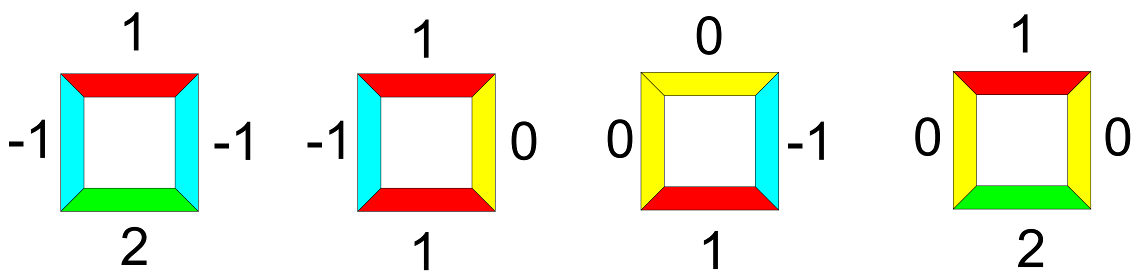
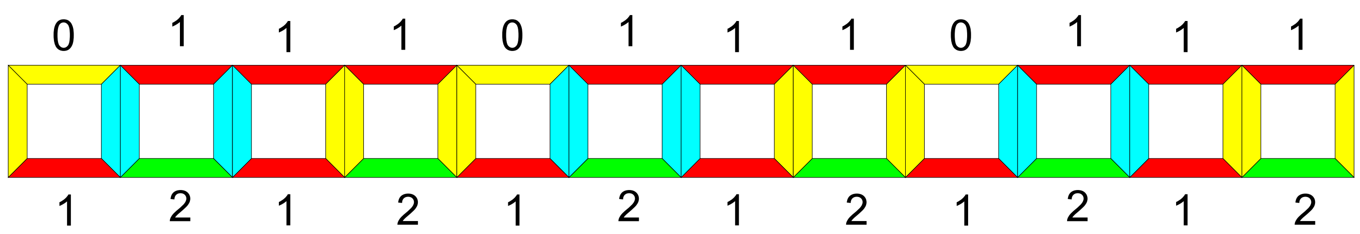
B(\sqrt{2}) &= \ldots 1\ 1\ 2\ 1\ 2\ 1\ 2\ 1\ 1\ 2\ 1\ 1\ \ldots
\end{aligned}$$

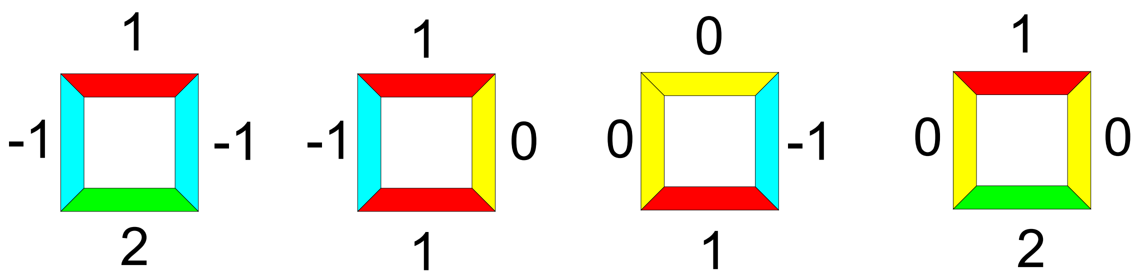


The first tile set T_2 is designed so that it admits a tiling of every infinite horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(2\alpha)$, for all $\alpha \in \mathbb{R}$ satisfying

$$\begin{array}{ccccc} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}$$

For example, with $\alpha = \frac{3}{4}$:





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$$\left.\begin{array}{ccccc} 0 & \leq & \alpha & \leq & 1, \text{ and} \\ 1 & \leq & 2\alpha & \leq & 2. \end{array}\right\} \iff \frac{1}{2} \leq \alpha \leq 1$$

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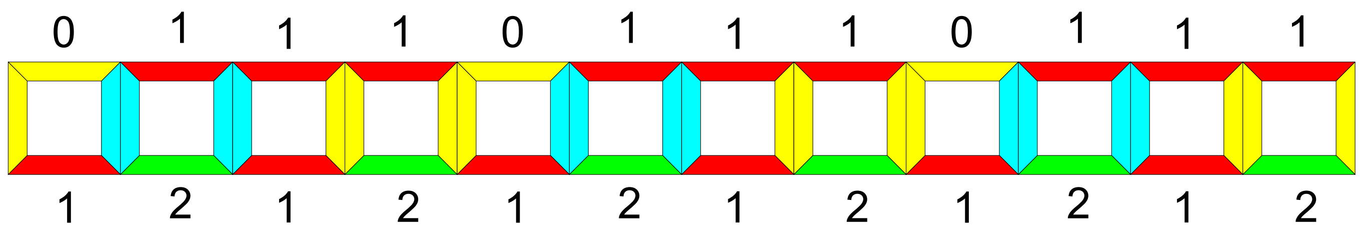
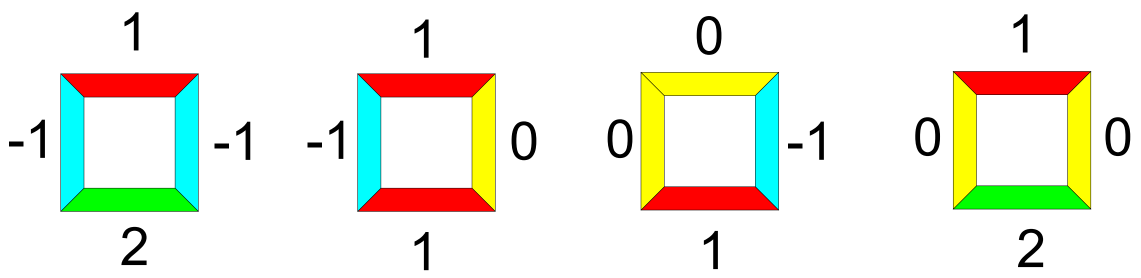
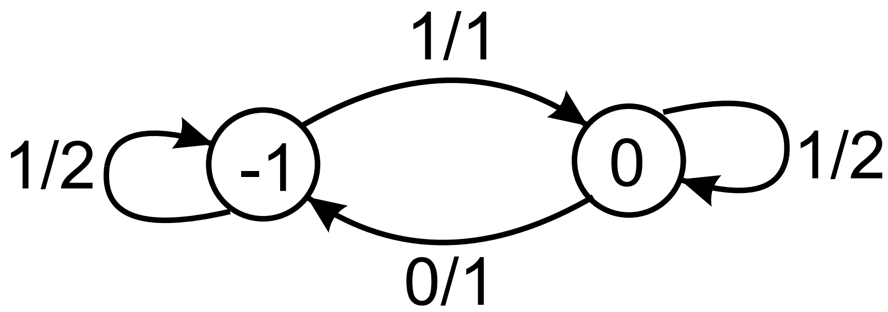
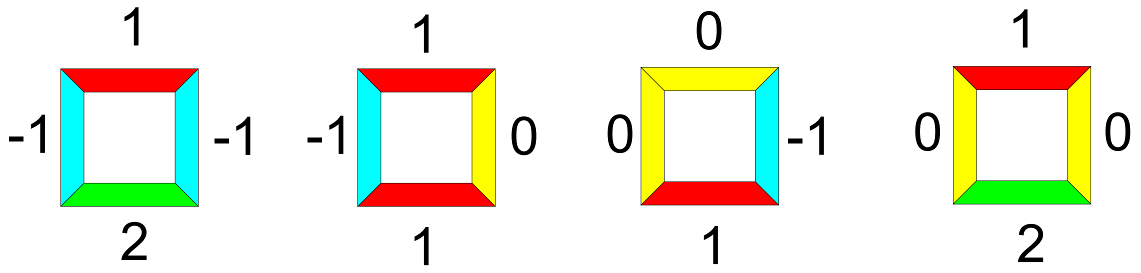




The four tiles can be also interpreted as transitions of a finite state transducer whose states are the vertical colors and input/output symbols of transitions are the top and the bottom colors:



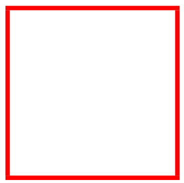
A tiling of an infinite horizontal strip is a bi-infinite path whose input symbols and output symbols read the top and bottom colors of the strip. We must have enough transitions to allow the transducer to convert $B(\alpha)$ into $B(2\alpha)$.



This is guaranteed by including in the tile set for every $\frac{1}{2} \leq \alpha \leq 1$ and every $k \in \mathbb{Z}$ the following tile

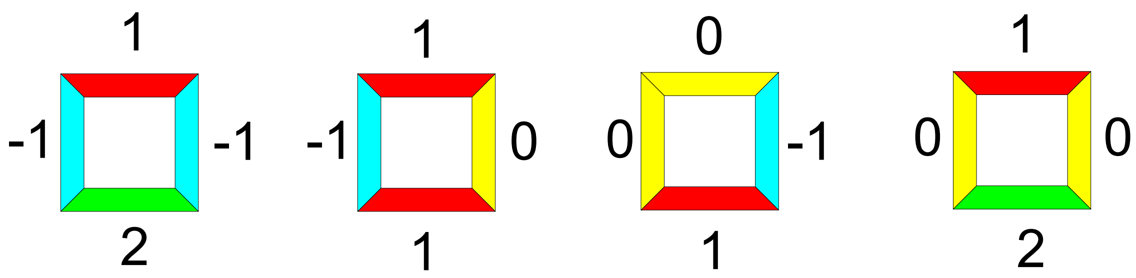
$$B_k(\alpha)$$

$$2\lfloor (k-1)\alpha \rfloor - \lfloor 2(k-1)\alpha \rfloor$$



$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$



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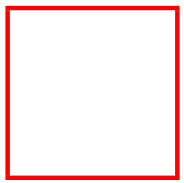
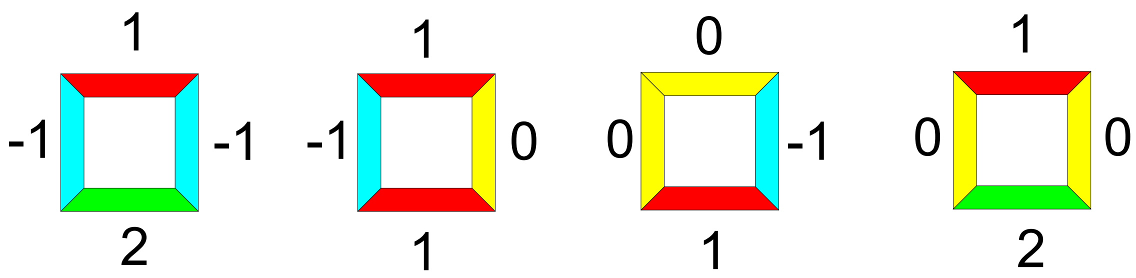


$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

(1) For fixed α the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of α and 2α, respectively.



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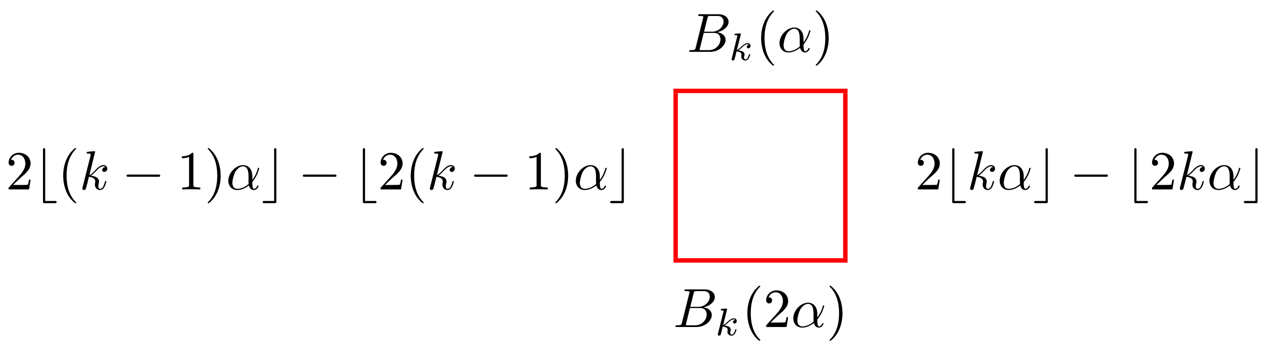
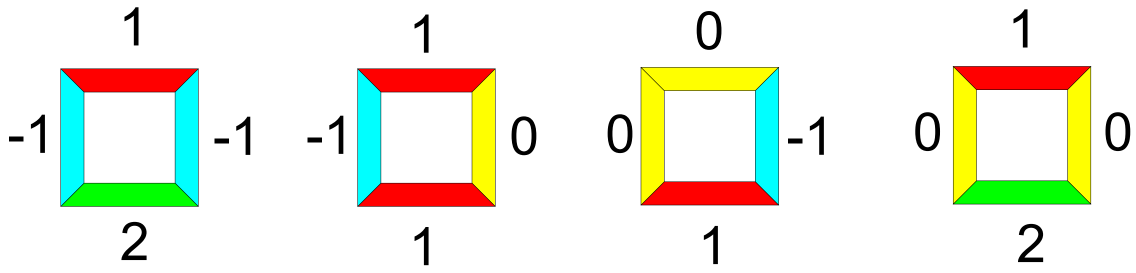


(2) A direct calculation shows that the tile multiplies by 2, that is,

$$2n + w = s + e.$$



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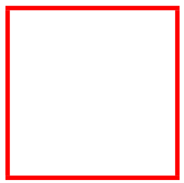

$$2\lfloor k\alpha \rfloor - \lfloor 2k\alpha \rfloor$$

$$B_k(2\alpha)$$

(3) There are only finitely many such tiles, even though there are infinitely many $k \in \mathbb{Z}$ and α. The tiles are the four tiles of T_2.

An analogous construction can be done for any rational multiplier q. We can construct the following tiles for all $k \in \mathbb{Z}$ and all α in the domain interval:

$$B_k(\alpha)$$

$$q\lfloor (k-1)\alpha \rfloor - \lfloor q(k-1)\alpha \rfloor \qquad\qquad q\lfloor k\alpha \rfloor - \lfloor qk\alpha \rfloor$$

$$B_k(q\alpha)$$

If q is a rational number and the domain interval is a finite interval then there are only a finite number of such tiles. The tiles multiply by q, and they admit a tiling of a horizontal strip whose top and bottom labels read $B(\alpha)$ and $B(q\alpha)$.

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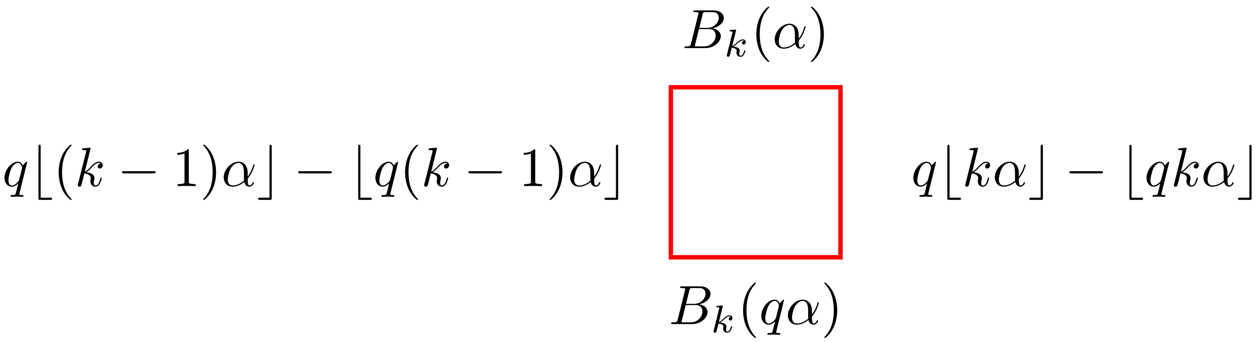

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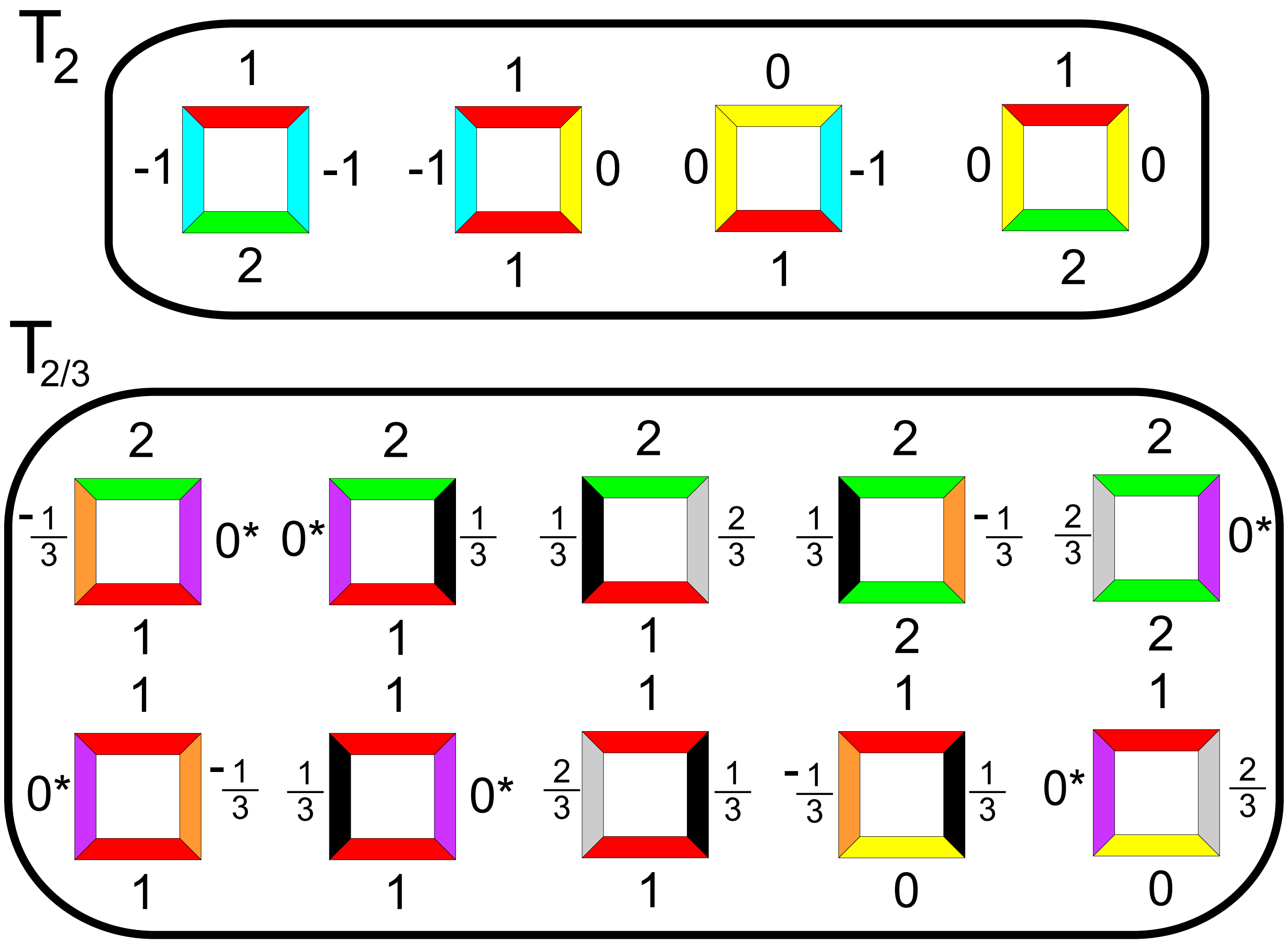
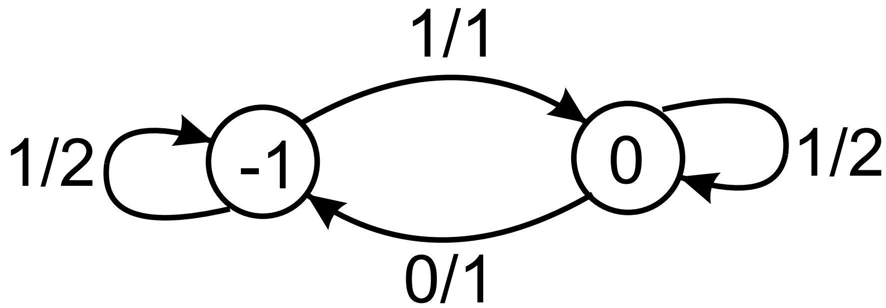
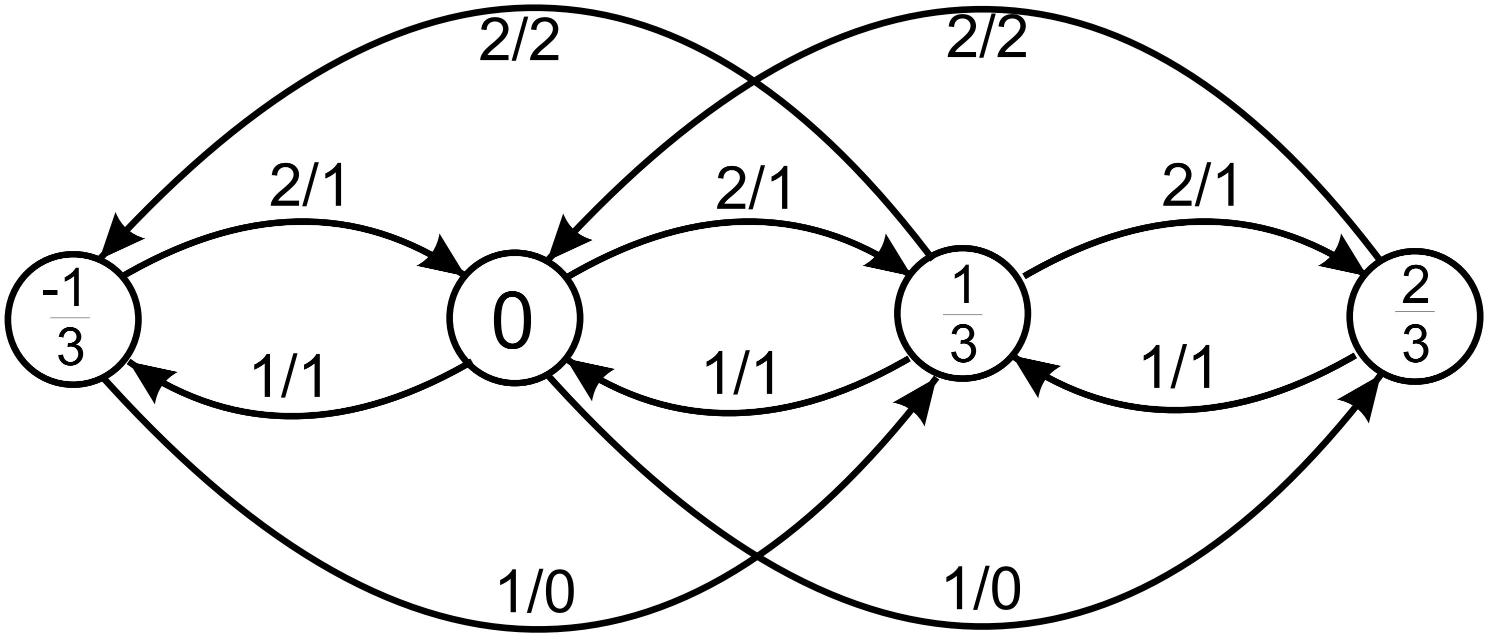
Our second tile set $T_{2/3}$ was constructed in this way for $q = \frac{2}{3}$ and $1 \leq \alpha \leq 2$.



T_2
$T_{2/3}$


1/1
1/2
-1
0
1/2
0/1

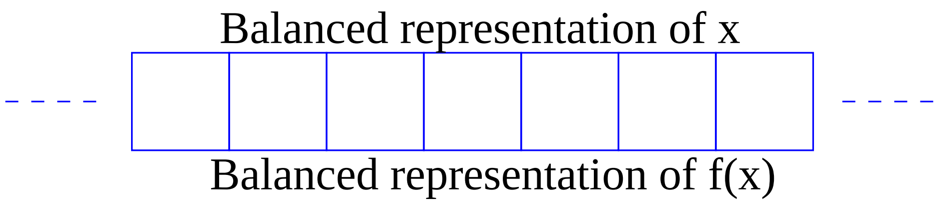

2/2
2/2
2/1
2/1
2/1
-1/3
0
1/3
2/3
1/1
1/1
1/1
1/0
1/0

Now we can see that the tiles admit valid tilings of the plane that simulate iterations of the piecewise linear dynamical system

$$f : [\frac{1}{2}, 2] \longrightarrow [\frac{1}{2}, 2]$$

where

$$f(x) = \begin{cases} 2x, & \text{if } x \leq 1, \text{ and} \\ \frac{2}{3}x, & \text{if } x > 1. \end{cases}$$



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Balanced representation of x

Balanced representation of $f^2(x)$

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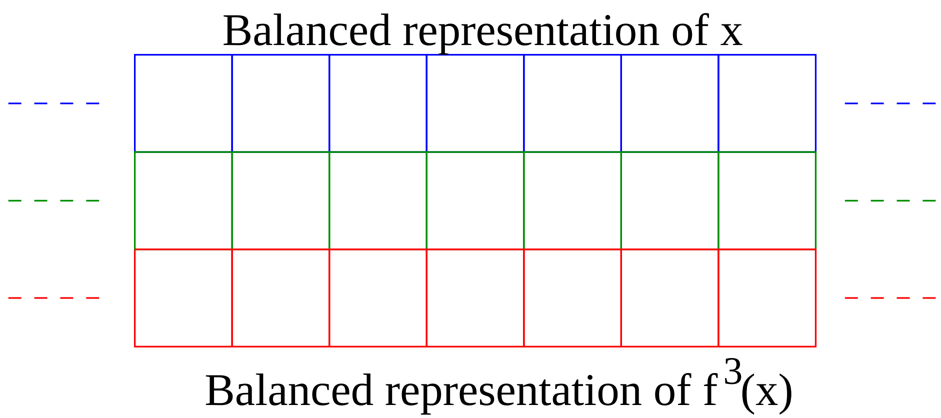

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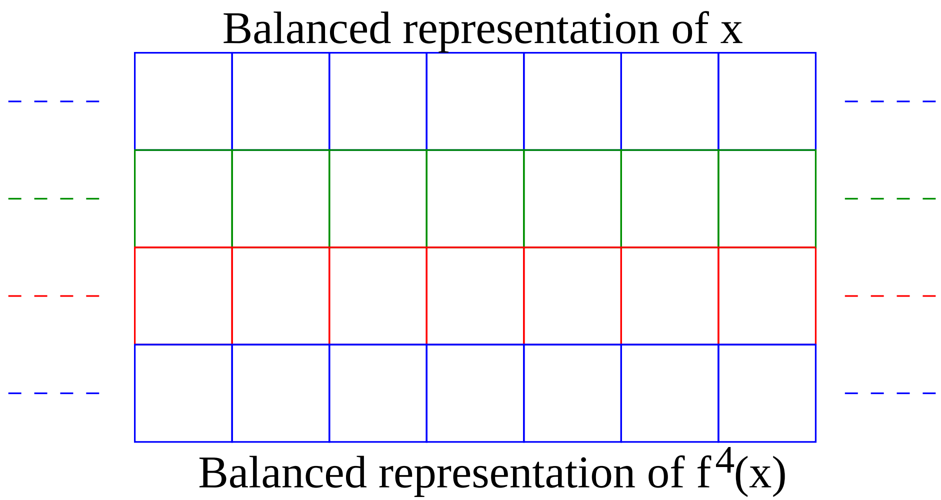


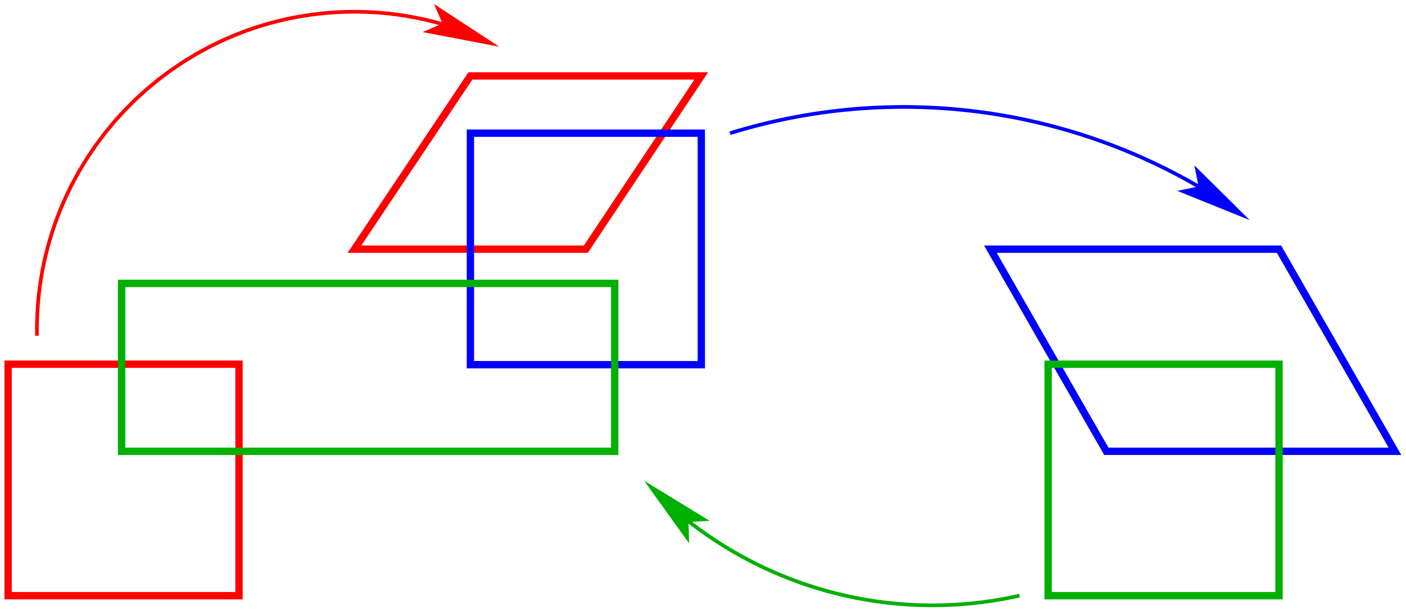
Similar construction can be effectively carried out for any piecewise linear function on a union of finite intervals of $\mathbb{R}$, as long as the multiplications are with rational numbers q.

In order to prove undecidability results concerning tilings it is desirable to simulate slightly more complex dynamical systems that can carry out Turing computations.

We generalize the construction in two ways:

- from linear maps to affine maps, and
- from $\mathbb{R}$ to $\mathbb{R}^2$, (or $\mathbb{R}^d$ for any d).

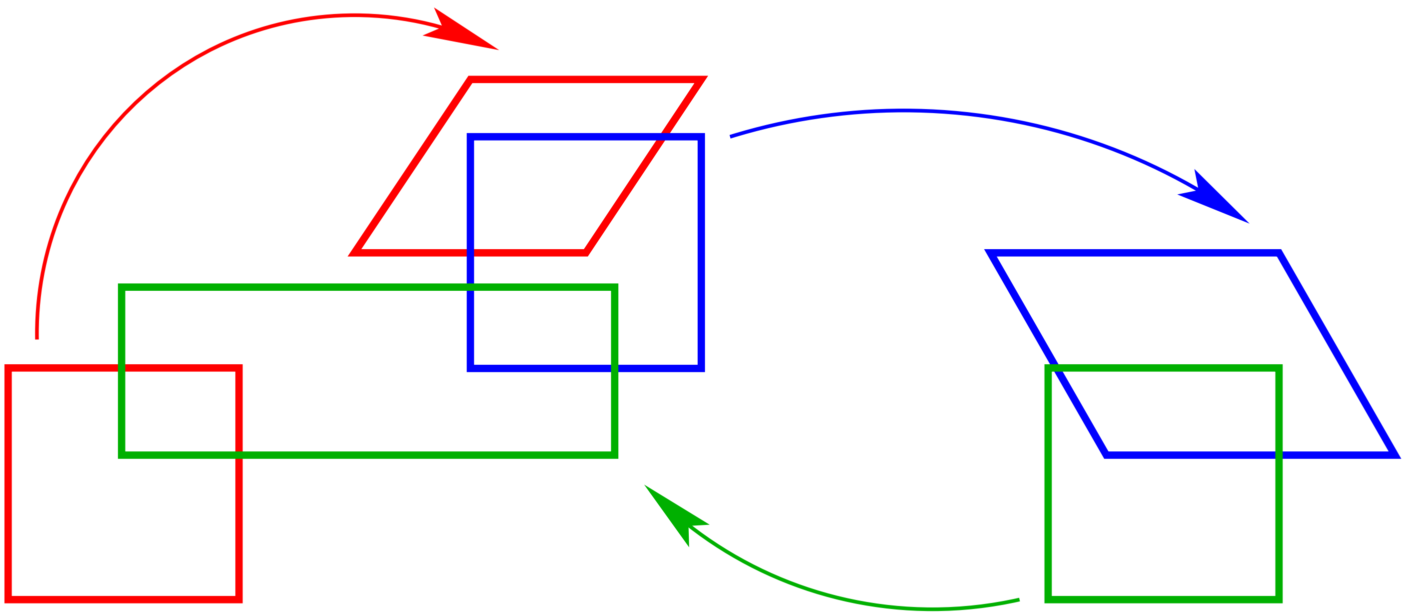
Immortality of piecewise affine maps



Consider a system of finitely many pairs (U_i, f_i) where

- U_i are disjoint unit squares of the plane with integer corners,
- f_i are affine transformations with rational coefficients.

Square U_i is understood as the domain where f_i may be applied.



The system determines a function

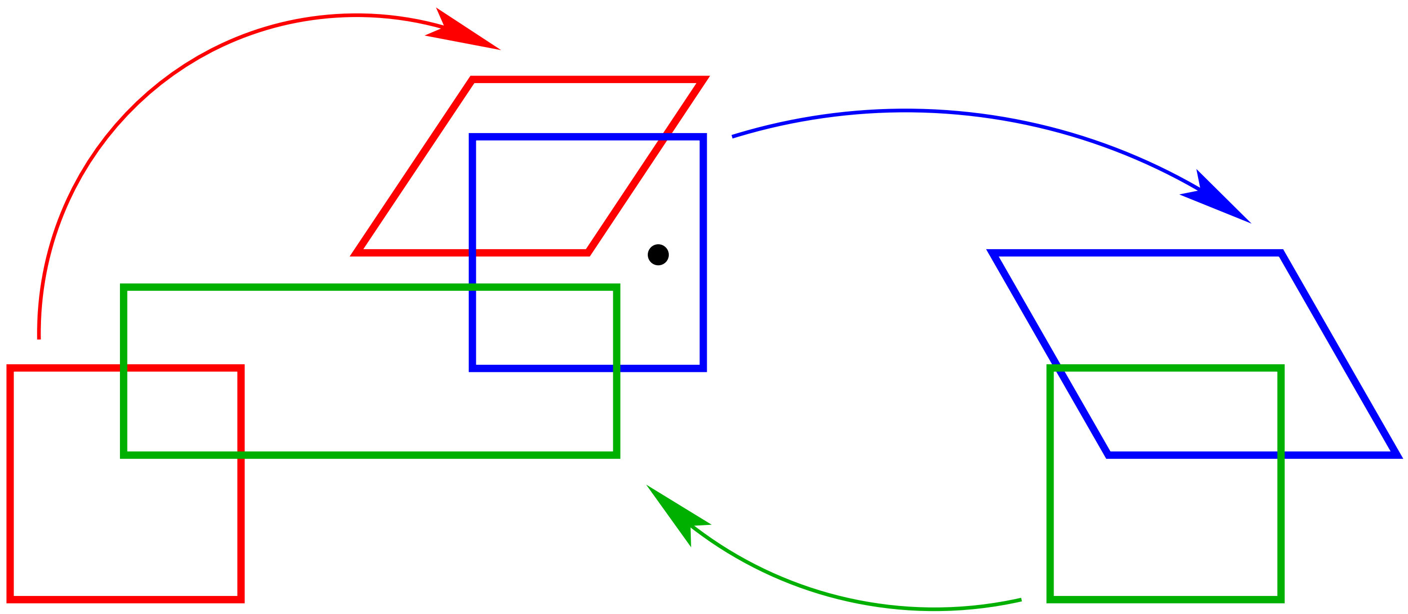
$$f : D \longrightarrow \mathbb{R}^2$$

whose domain is

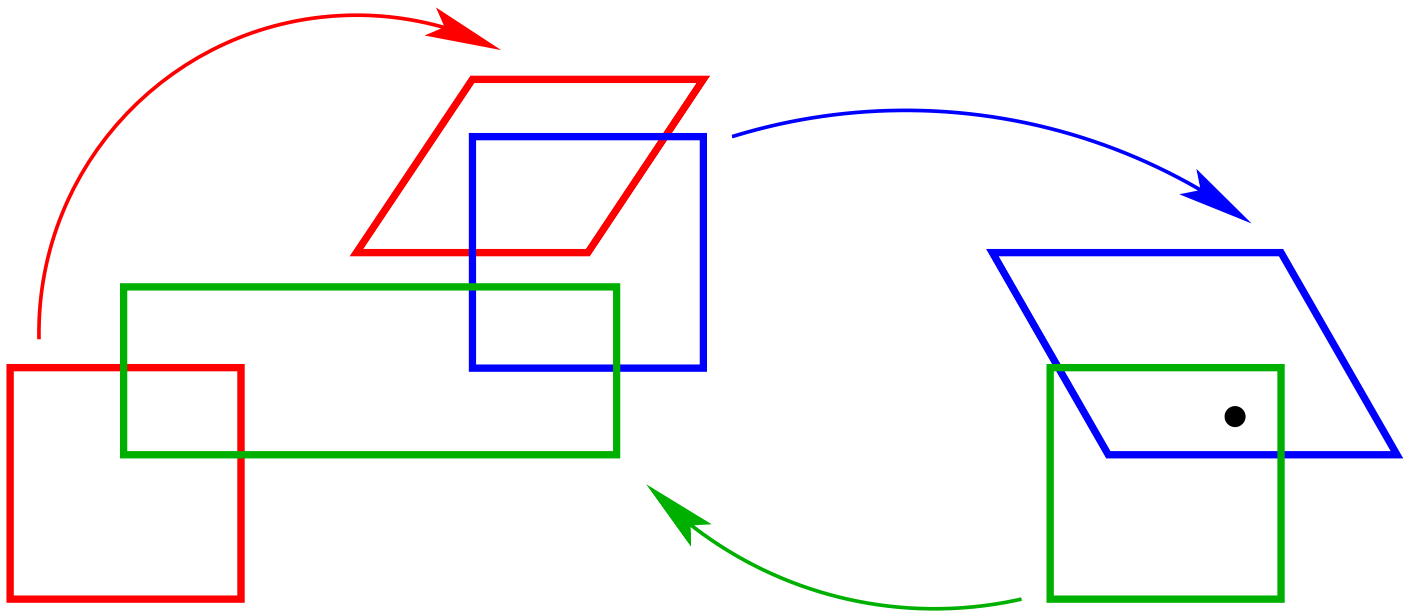
$$D = \bigcup_i U_i$$

and

$$f(\vec{x}) = f_i(\vec{x}) \text{ for all } \vec{x} \in U_i.$$



The orbit of $\vec{x} \in D$ is the iteration of f starting at point $\vec{x}$. The iteration can be continued as long as the point remains in the domain D.



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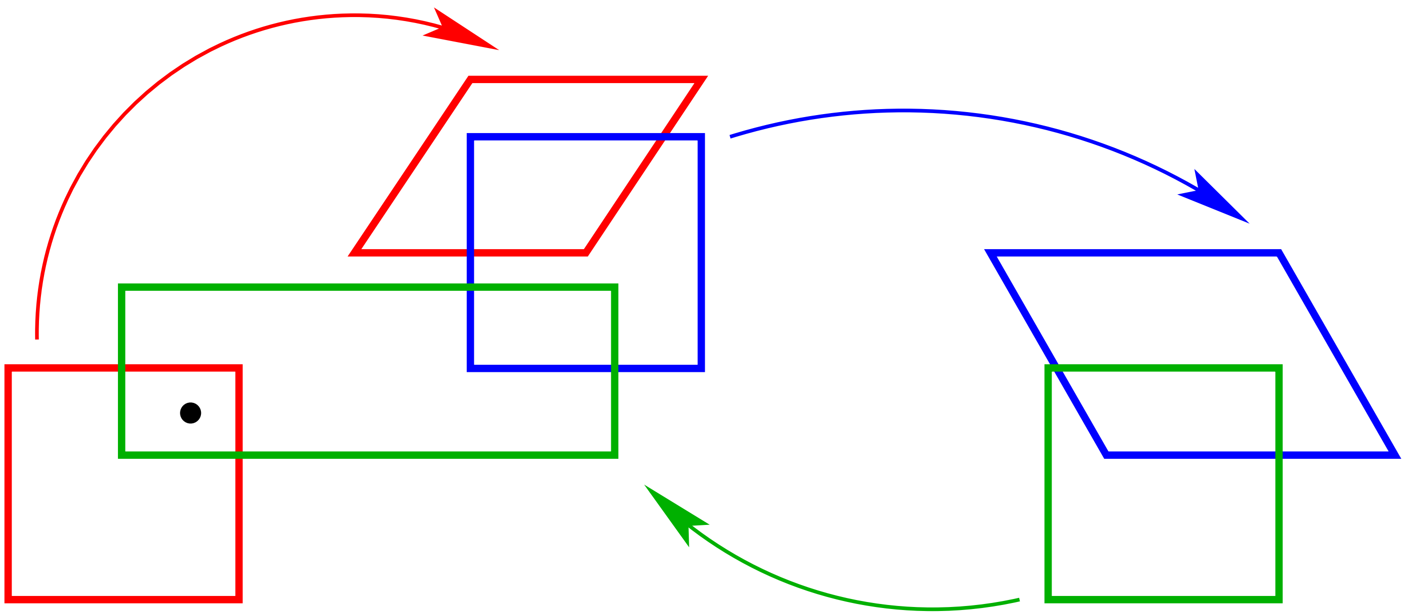

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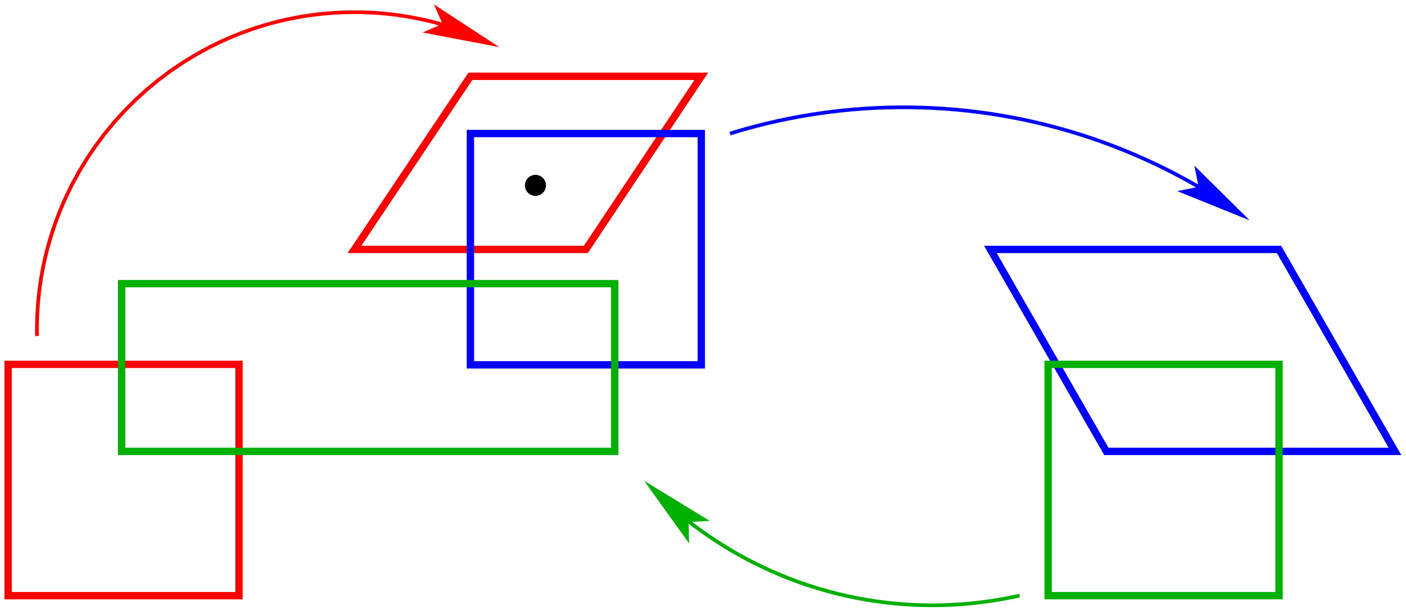

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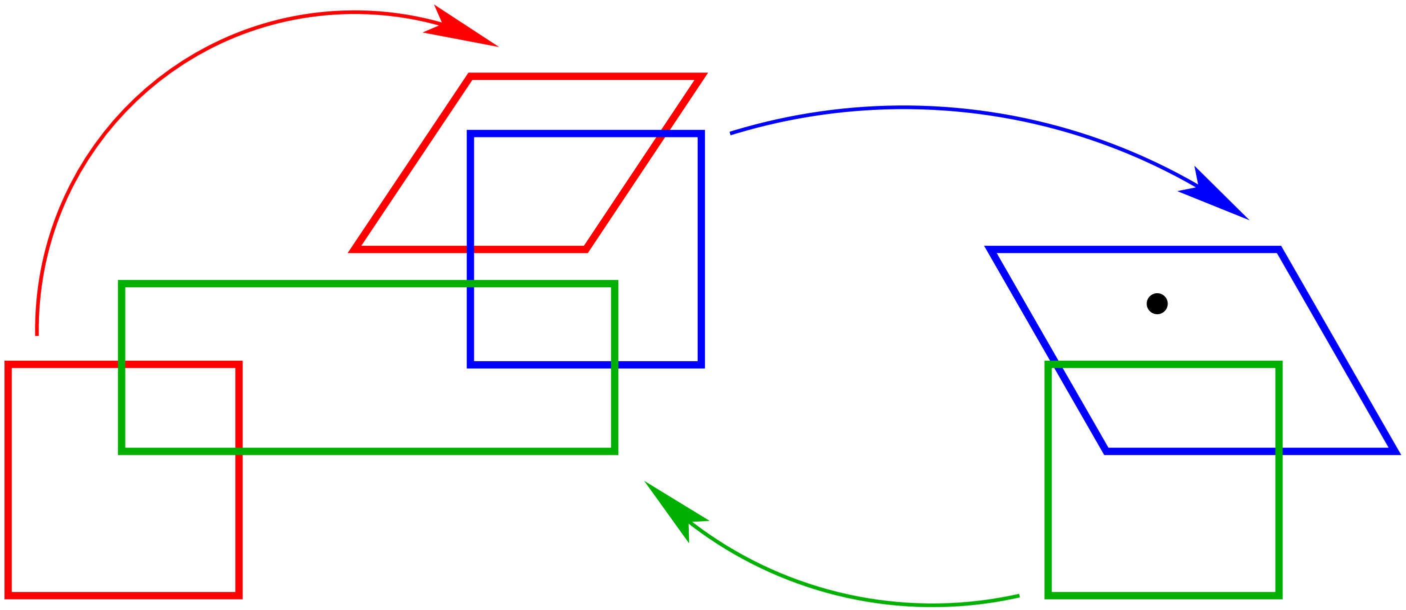
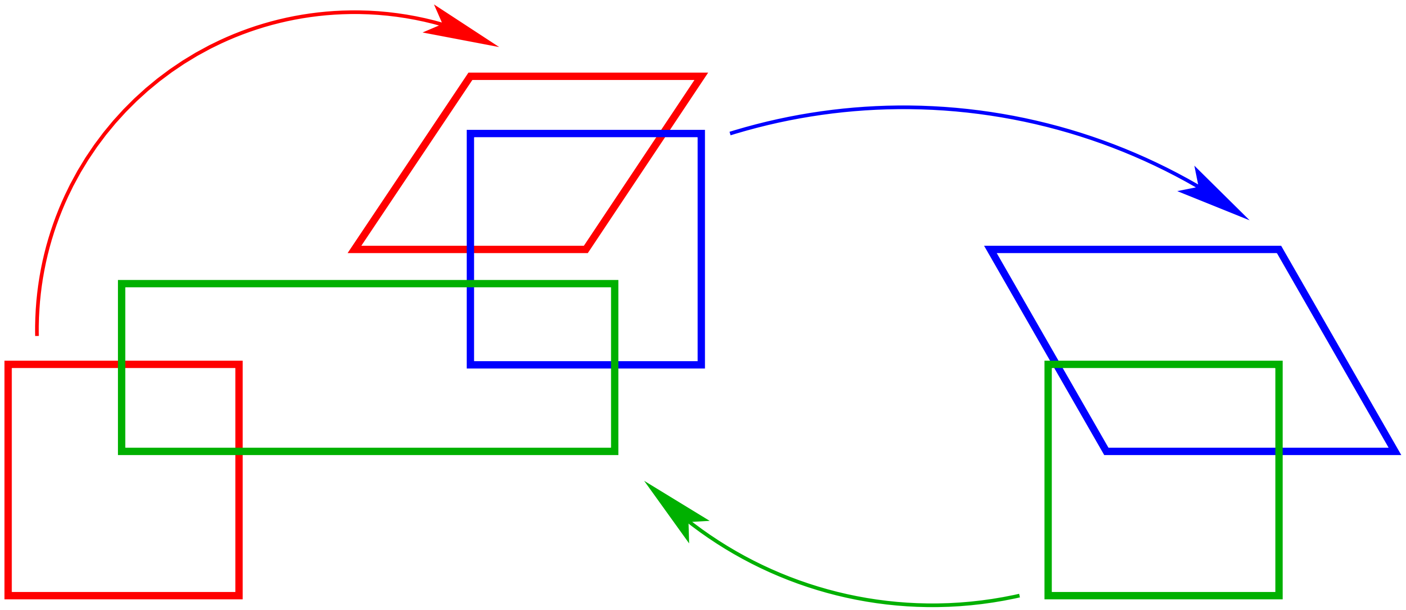


But if the point goes outside of the domain, the system **halts**.

If the iteration always halts, regardless of the starting point $\vec{x}$, the system is **mortal**. Otherwise it is **immortal**: there is an immortal point $\vec{x} \in D$ from which a non-halting orbit begins.

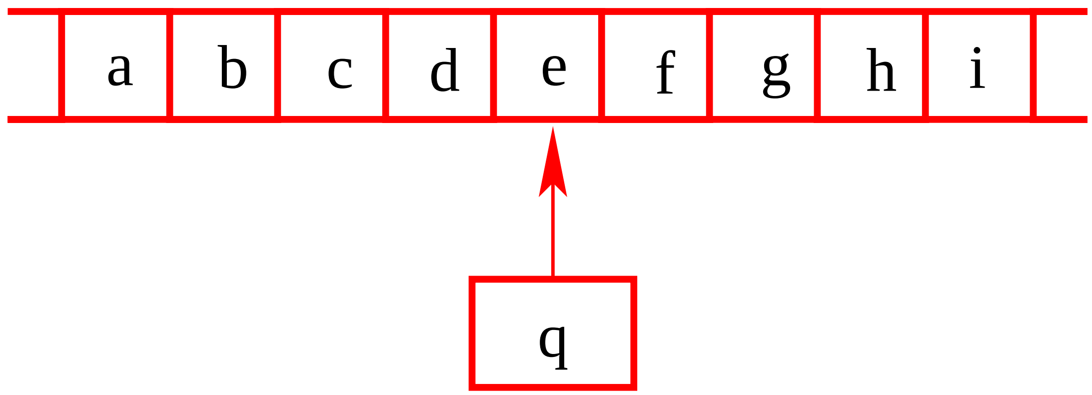


Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.

To prove the undecidability one can use a standard technique for transforming **Turing machines** into two-dimensional piecewise affine transformations.

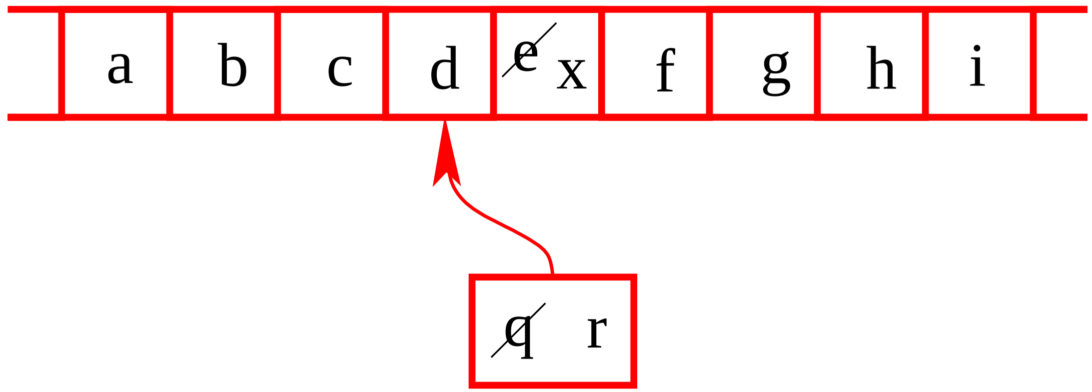
Turing machine configuration



is encoded as the pair $(x, y) \in \mathbb{R}^2$ where the digits of x and y (in some suitably large base B) express the contents of the left and right halves of the tape:

$$\begin{cases} x &= ef.ghi\ldots \\ y &= qd.cba\ldots \end{cases}$$

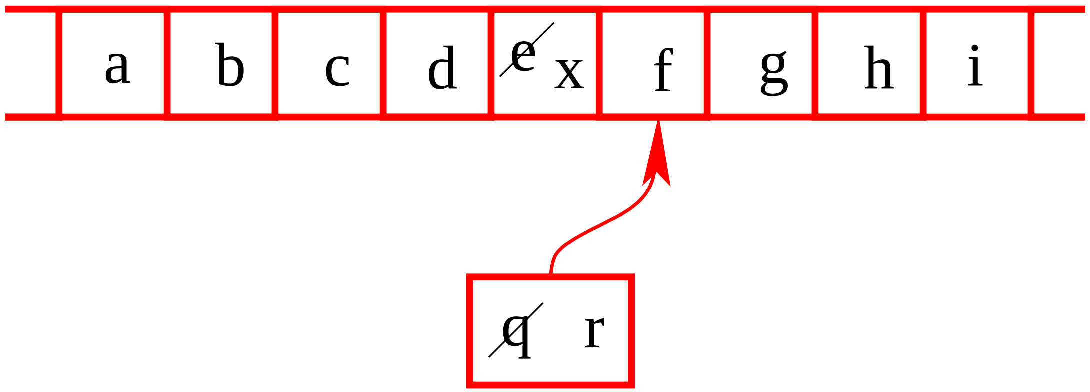
The integer parts of x and y determine the next move of the machine, that is, the next move depends on the integer unit square containing point (x, y).



A left move of the Turing machine requires that the digits of x and y are shifted one position to the right and left, respectively. Adding suitable (integer) constants takes care of changes in the state q and the current tape symbol e.

$$\left\{\begin{array}{lll} x & = & \mathbf{e}f.ghi\ldots \\ y & = & \mathbf{qd}.cba\ldots \end{array}\right. \qquad \mapsto \qquad \left\{\begin{array}{lll} x' & = & \mathbf{dx}.fghi\ldots \\ y' & = & \mathbf{r}c.ba\ldots \end{array}\right.$$

This is an affine transformation whose matrix is $\begin{pmatrix} \frac{1}{B} & 0 \\ 0 & B \end{pmatrix}$.



Analogously, a right move is simulated by an affine transformation whose matrix is

$$\begin{pmatrix} B & 0 \\ 0 & \frac{1}{B} \end{pmatrix}.$$

Additional changes in the integer parts complete the transformation:

$$\left\{ \begin{array}{lcl} x & = & \mathbf{e}f.ghi\ldots \\ y & = & \mathbf{q}d.cba\ldots \end{array} \right. \qquad \mapsto \qquad \left\{ \begin{array}{lcl} x' & = & fg.hi\ldots \\ y' & = & \mathbf{rx}.dcba\ldots \end{array} \right.$$

A given Turing machine is converted in this way into a system of unit squares U_i and corresponding affine transformations f_i. Then iterations of the Turing machine on arbitrary configurations correspond to iterations of the affine maps.

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In particular, the system of affine maps has an immortal point if and only if the Turing machine has an **immortal configuration,** that is, a configuration that leads to a non-halting computation in the Turing machine. But we have the following result:

Theorem (Hooper 1966): It is undecidable if a given Turing machine has any immortal configurations.

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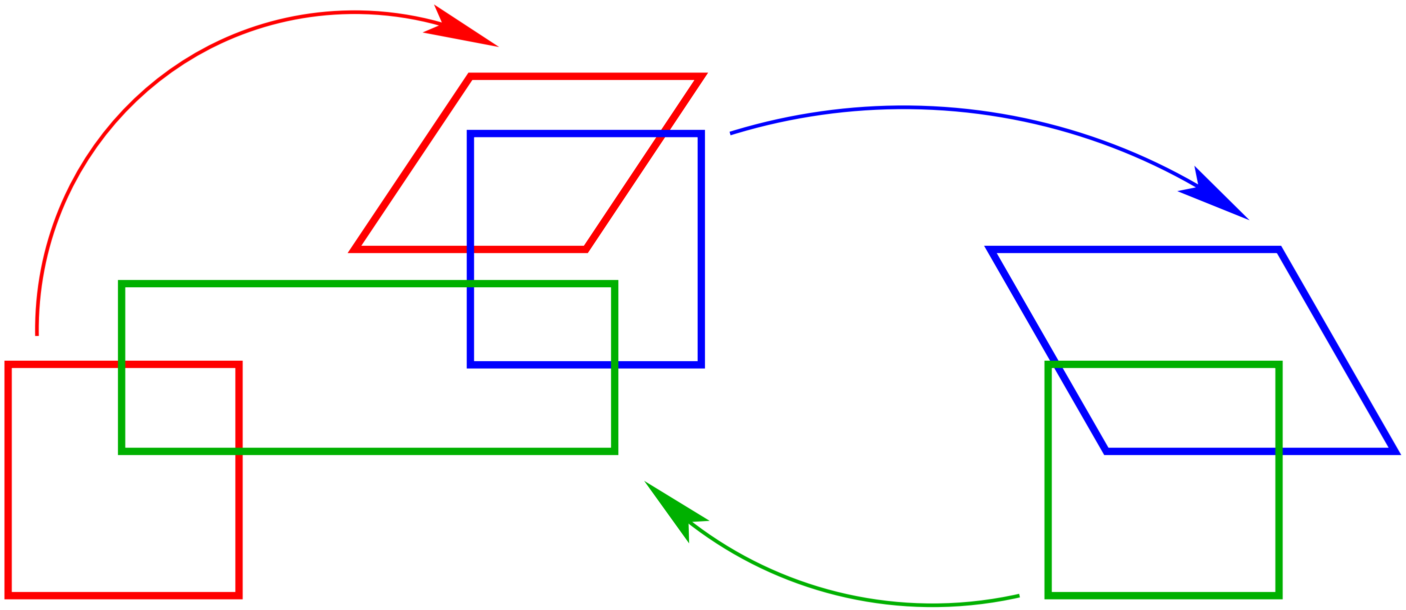
(Interesting historical note: Hooper and Berger were both students of Hao Wang, at the same time. Their results are of same flavor but the proofs are independent.)



Immortality problem: Is a given system of affine maps immortal?

Proposition: The immortality problem is undecidable.

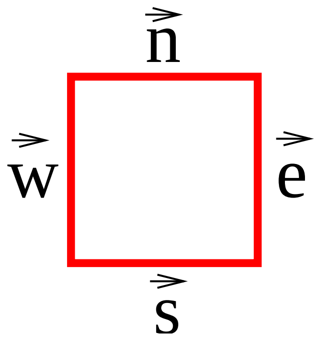
The proposition now follows from Hooper's theorem. $\square$

Next we reduce the immortality problem to the tiling problem, by effectively constructing Wang tiles that are forced to simulate iterations of the given piecewise affine maps. Then a valid tiling of the plane exists if and only if the dynamical system has an infinite orbit, i.e. is not mortal.

The construction is very similar to the earlier construction of 14 aperiodic tiles.

The colors in our Wang tiles are elements of $\mathbb{R}^2$.

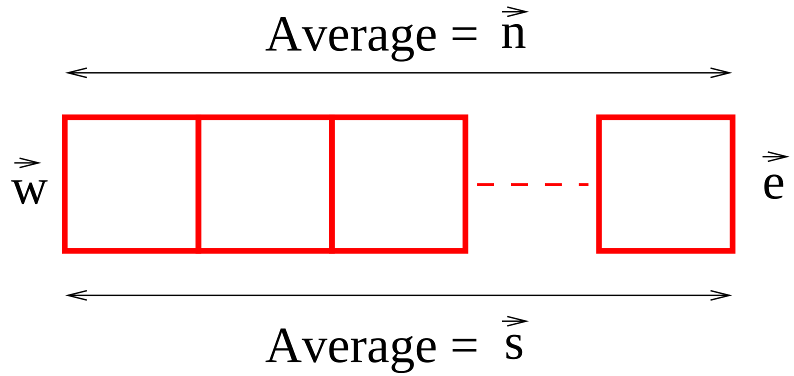
Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an affine function. We say that tile



computes function f if

$$f(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

Suppose we have a correctly tiled horizontal segment of length n where all tiles compute the same f.



It easily follows that

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

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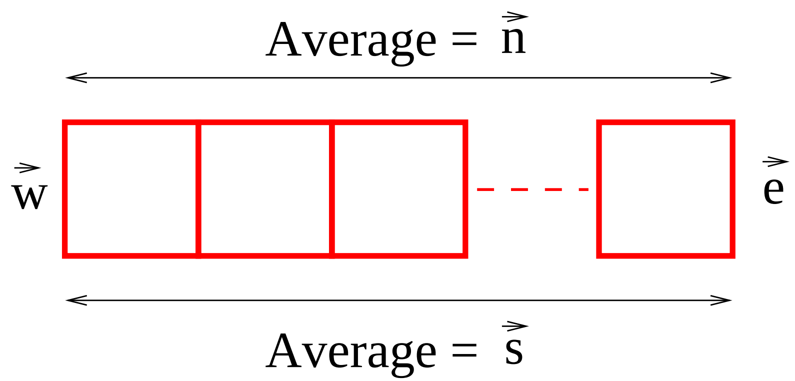

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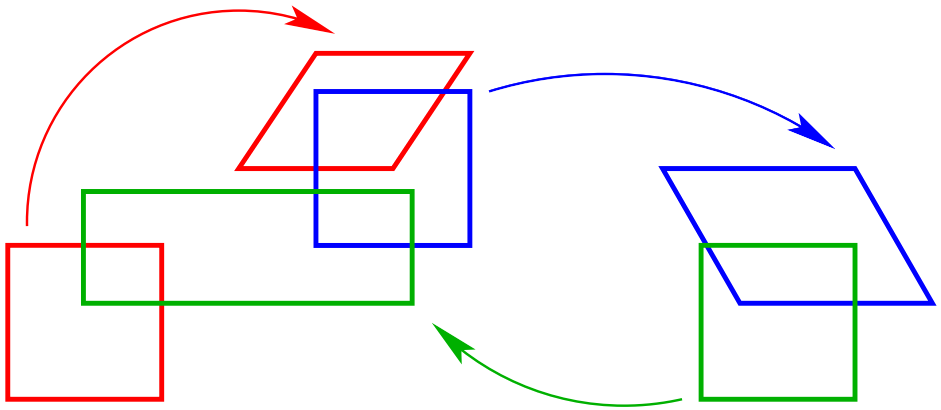
where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

As the segment is made longer, the effect of the carry in and out labels $\vec{w}$ and $\vec{e}$ vanish.



Consider a system of affine maps f_i and unit squares U_i.

For each i we construct a set T_i of Wang tiles

- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

An additional label i on the vertical edges makes sure that tiles of different sets T_i and T_j cannot be mixed on any horizontal row of tiles. Let

$$T = \bigcup_i T_i.$$

Claim: If T admits a valid tiling then the system of affine maps has an immortal point.

Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

Claim: If T admits a valid tiling then the system of affine maps has an immortal point.

Indeed: An immortal point is obtained as the average of the top labels on a horizontal row of the tiling. The averages on subsequent horizontal rows below are the iterates of that point under the dynamical system.

If the average over an infinite horizontal row does not exist then we take an accumulation point of averages of finite segments instead... this always exists.

We still have to detail how to choose the tiles so that also the converse is true: any immortal orbit of the affine maps corresponds to a valid tiling.

For any $\vec{x} \in \mathbb{R}^2$ and $k \in \mathbb{Z}$ denote

$$B_k(\vec{x}) = \lfloor k\vec{x} \rfloor - \lfloor (k-1)\vec{x} \rfloor$$

where the floor is taken on both coordinates separately:

$$\lfloor (x, y) \rfloor = (\lfloor x \rfloor, \lfloor y \rfloor).$$

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The balanced (or sturmian) representation of vector $\vec{x}$ is the two-way infinite sequence

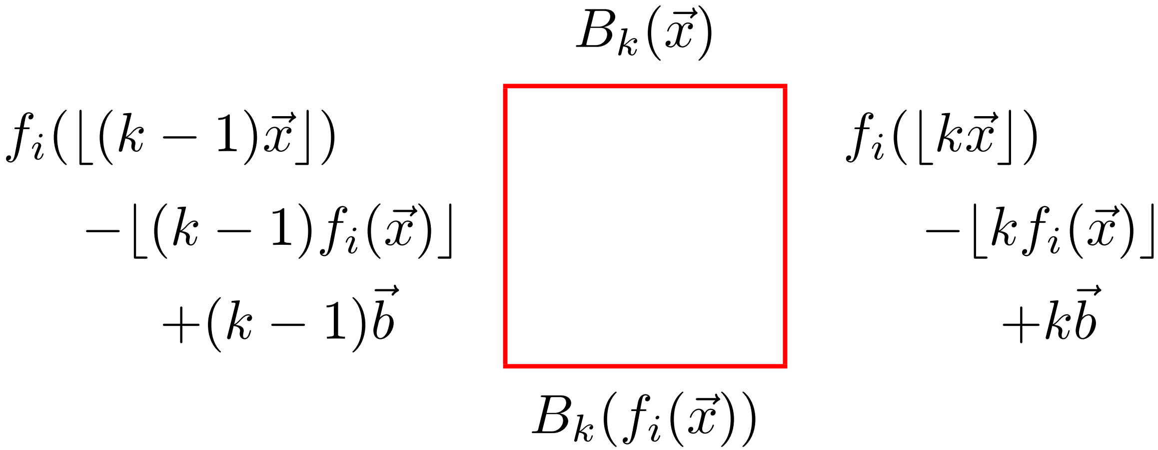
$$B(\vec{x}) = \ldots B_{-2}(\vec{x}), B_{-1}(\vec{x}), B_0(\vec{x}), B_1(\vec{x}), B_2(\vec{x}), \ldots$$

In other words, the sequence consists of the balanced representations of both coordinates of the vector.

The tile set corresponding to a rational affine map

$$f_i(\vec{x}) = M\vec{x} + \vec{b}$$

and its domain square U_i consists of all tiles



where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

$$\begin{array}{ccc} & B_k(\vec{x}) & \\ \begin{array}{r} f_i(\lfloor (k-1)\vec{x}\rfloor) \\ -\lfloor (k-1) f_i(\vec{x})\rfloor \\ +(k-1)\vec{b} \end{array} & \boxed{} & \begin{array}{r} f_i(\lfloor k\vec{x}\rfloor) \\ -\lfloor k f_i(\vec{x})\rfloor \\ +k\vec{b} \end{array} \\ & B_k(f_i(\vec{x})) & \end{array}$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(1) For fixed $\vec{x} \in U_i$ the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of $\vec{x}$ and $f_i(\vec{x})$, respectively.

$$
\begin{array}{ccc}
 & B_k(\vec{x}) & \\
\begin{array}{r} f_i(\lfloor (k-1)\vec{x} \rfloor) \\ -\lfloor (k-1) f_i(\vec{x}) \rfloor \\ +(k-1)\vec{b} \end{array} & \boxed{\phantom{\begin{array}{c} M \\ M \\ M \end{array}}} & \begin{array}{r} f_i(\lfloor k\vec{x} \rfloor) \\ -\lfloor k f_i(\vec{x}) \rfloor \\ +k\vec{b} \end{array} \\
 & B_k(f_i(\vec{x})) &
\end{array}
$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(2) A direct calculation shows that the tile computes function f_i, that is,

$$f_i(\vec{n}) + \vec{w} = \vec{s} + \vec{e}.$$

$$B_k(\vec{x})$$

$$\begin{array}{ccc}
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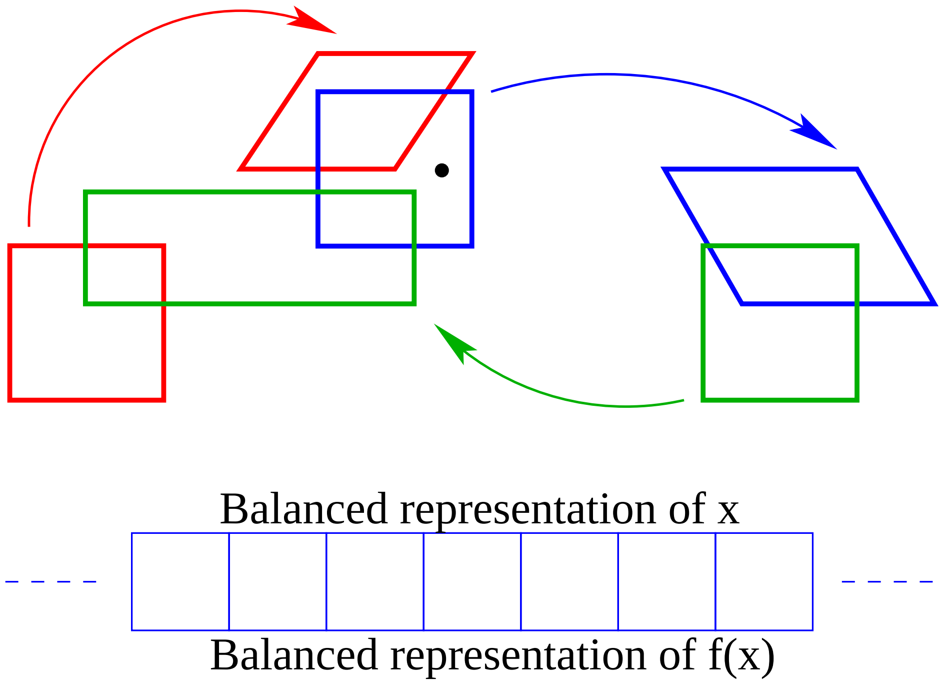
\end{array}$$

$$B_k(f_i(\vec{x}))$$

where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

(3) Because f_i is rational, there are only finitely many such tiles (even though there are infinitely many $k \in \mathbb{Z}$ and $\vec{x} \in U_i$). The tiles can be effectively constructed.

If there is an infinite orbit then a tiling exists where the labels of the horizontal rows read the balanced representations of the points of the orbit:



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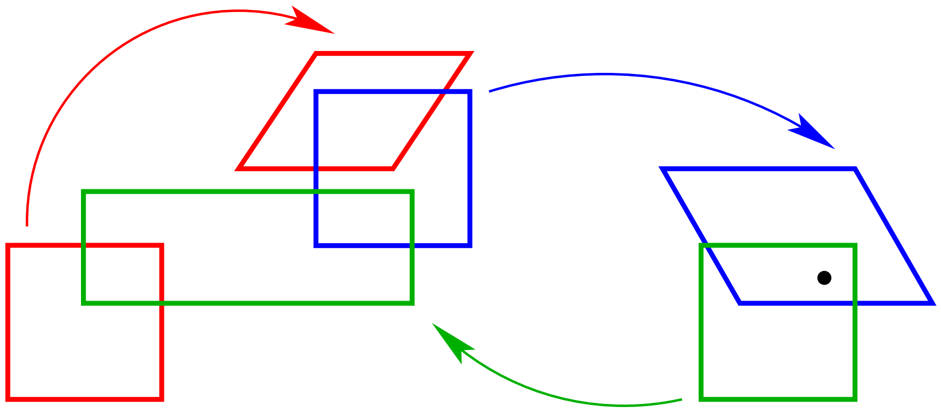

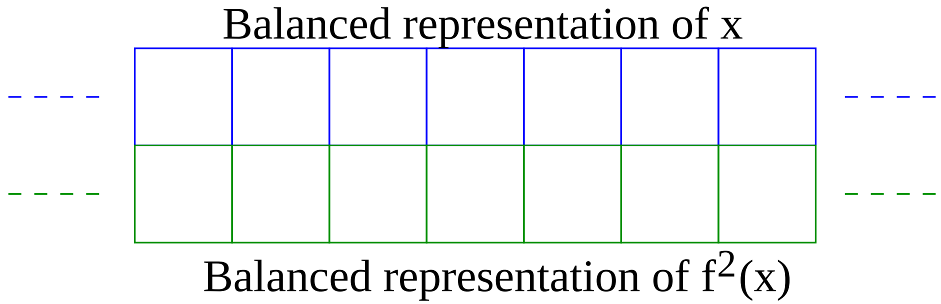

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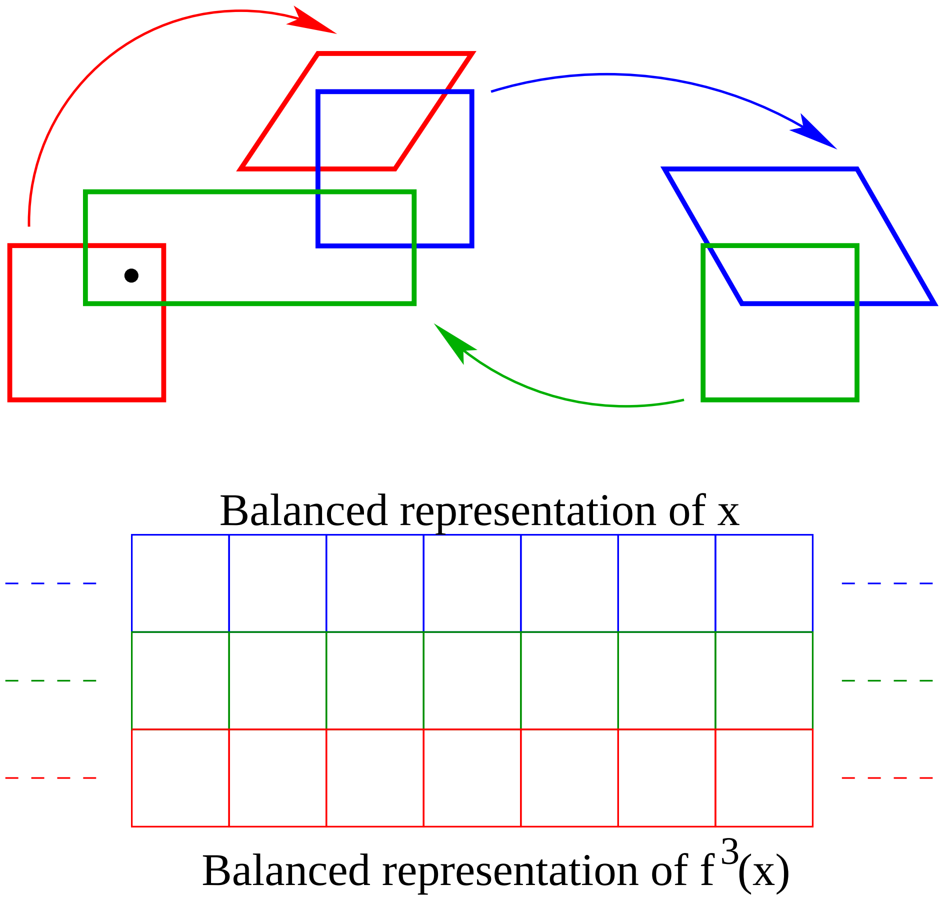

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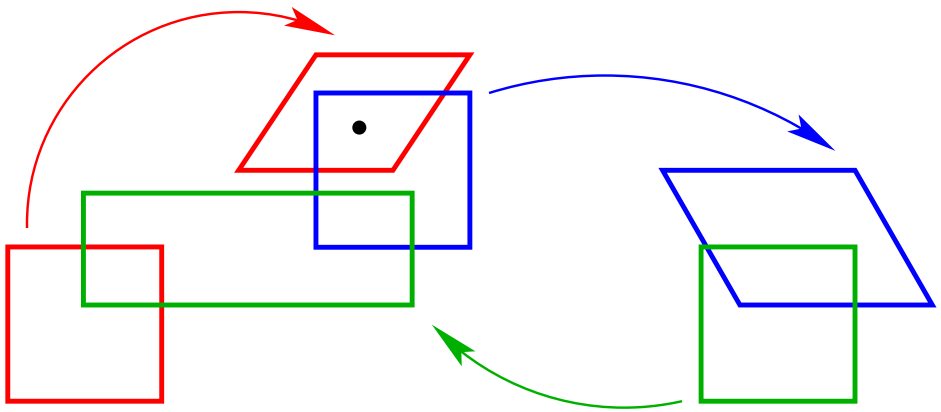

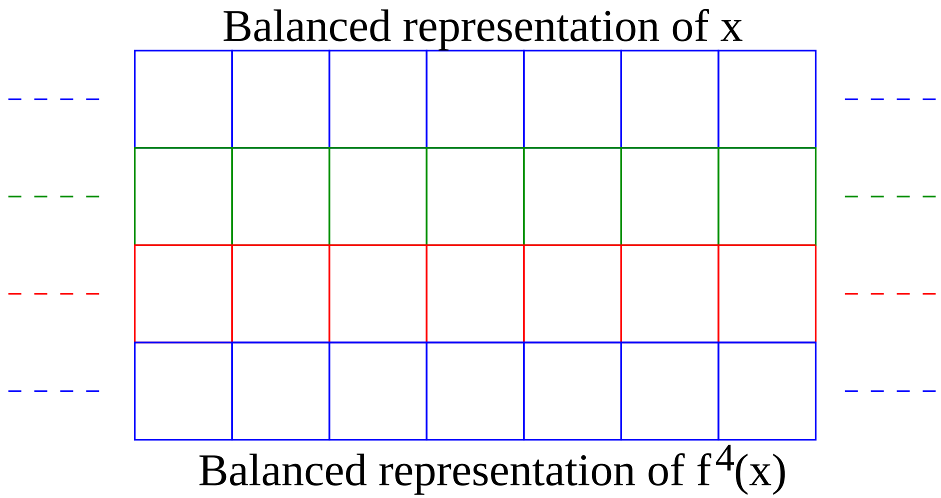

Conclusion: the tile set we constructed admits a tiling of the plane if and only if the system of affine maps is immortal. Undecidability of the tiling problem follows from the undecidability of the immortality problem. $\square$

The hyperbolic plane

The technique works well also in the hyperbolic plane. Hyperbolic plane is a plane where Euclid's fifth axiom does not hold: For any point P and a line L that does not contain P there are more than one lines through P that do not intersect L.

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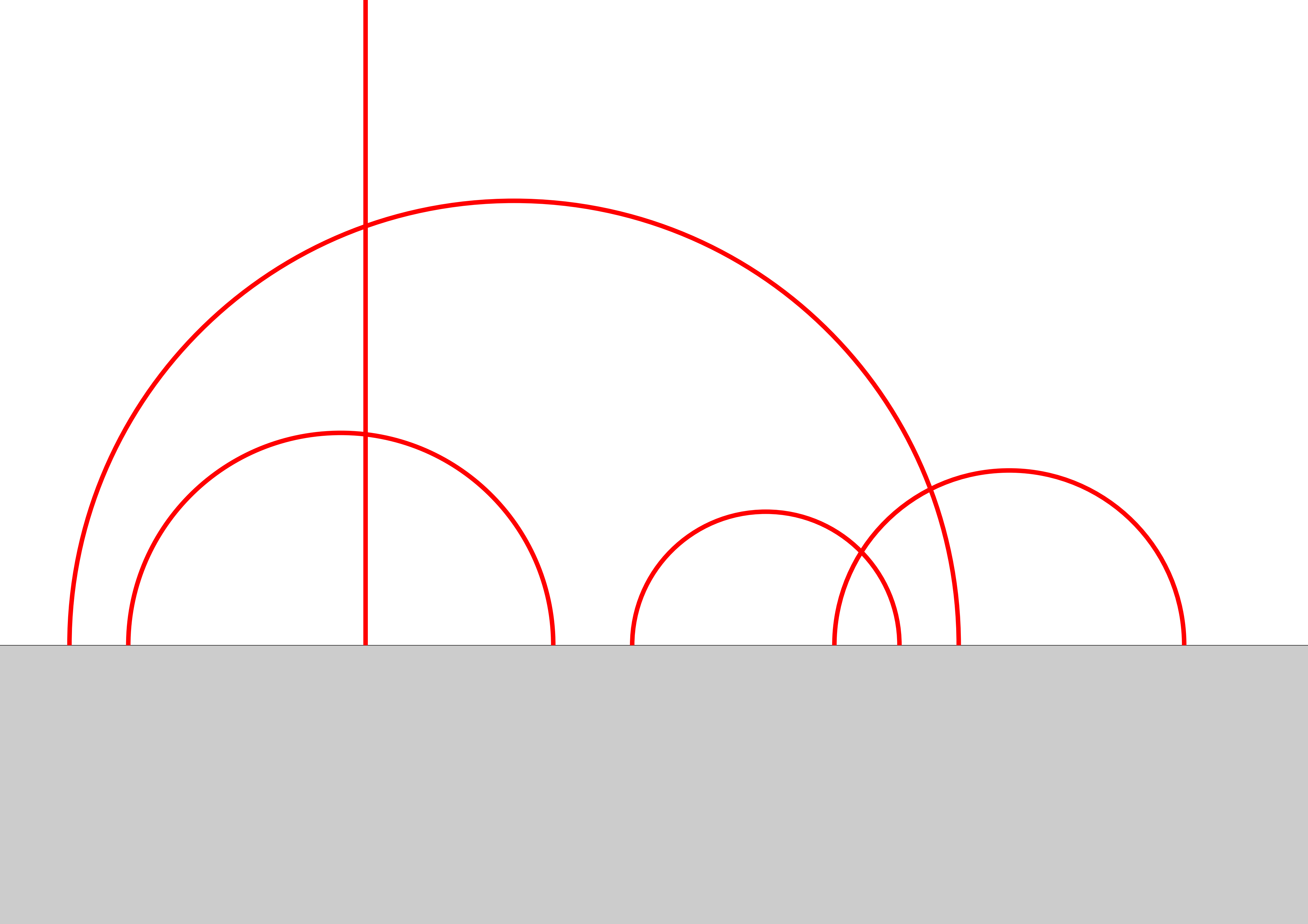
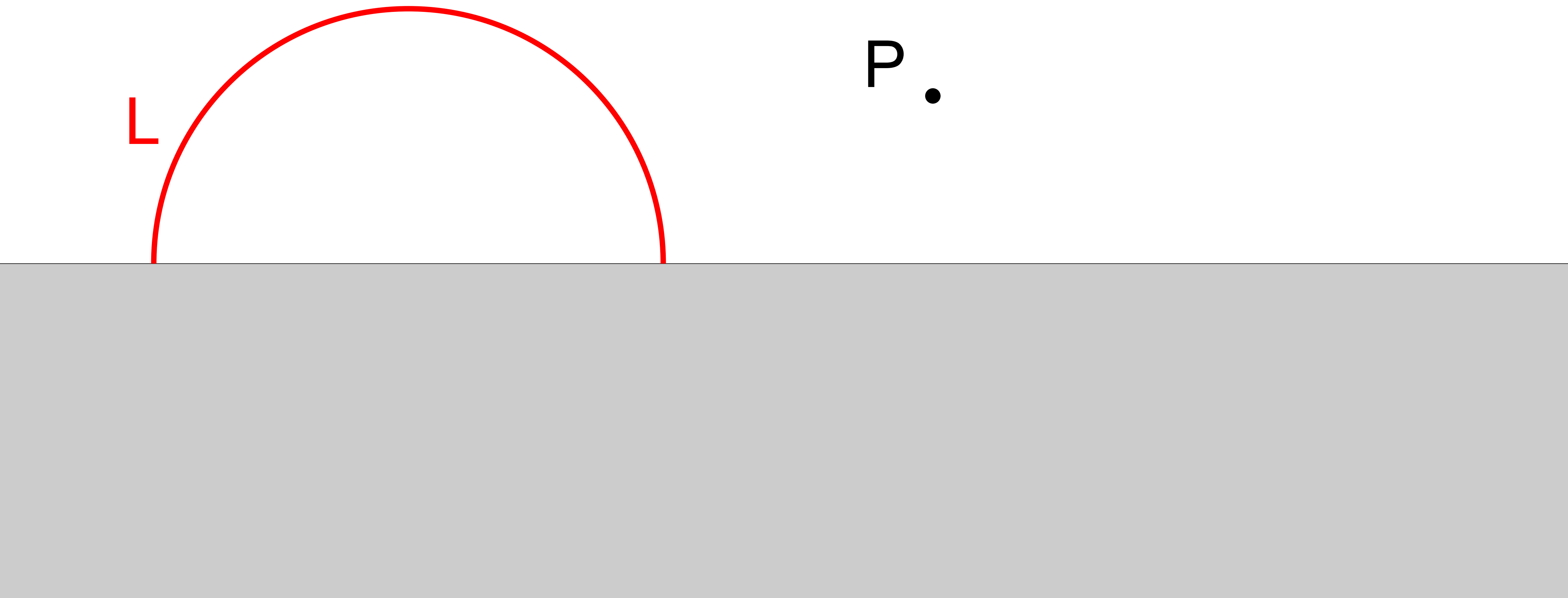
To display hyperbolic geometry on the screen (=Euclidean plane) we use the **half-plane** projection. The hyperbolic plane is represented as the Euclidean half plane. The division line is the **horizon**.

- Hyperbolic points are points in the open Euclidean half plane, and
- hyperbolic lines are semi-circles whose centers are on the horizon (and half-lines that are perpendicular to the horizon.)

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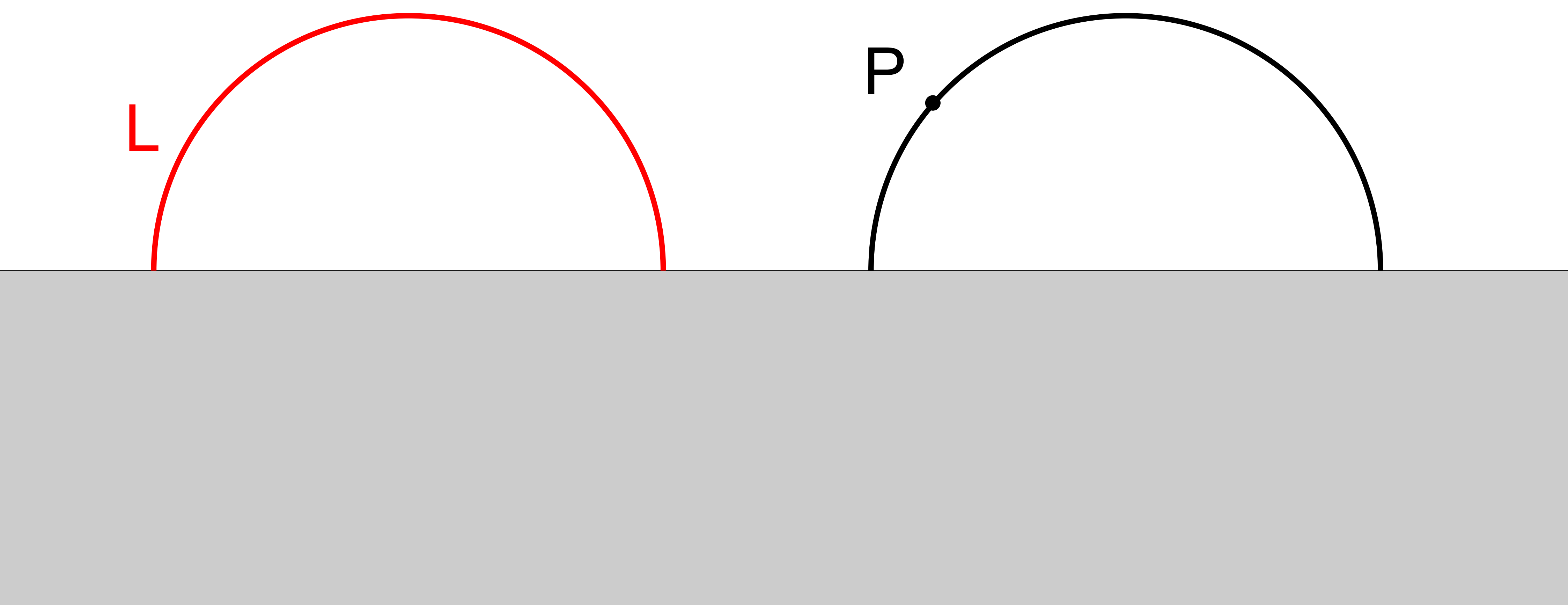
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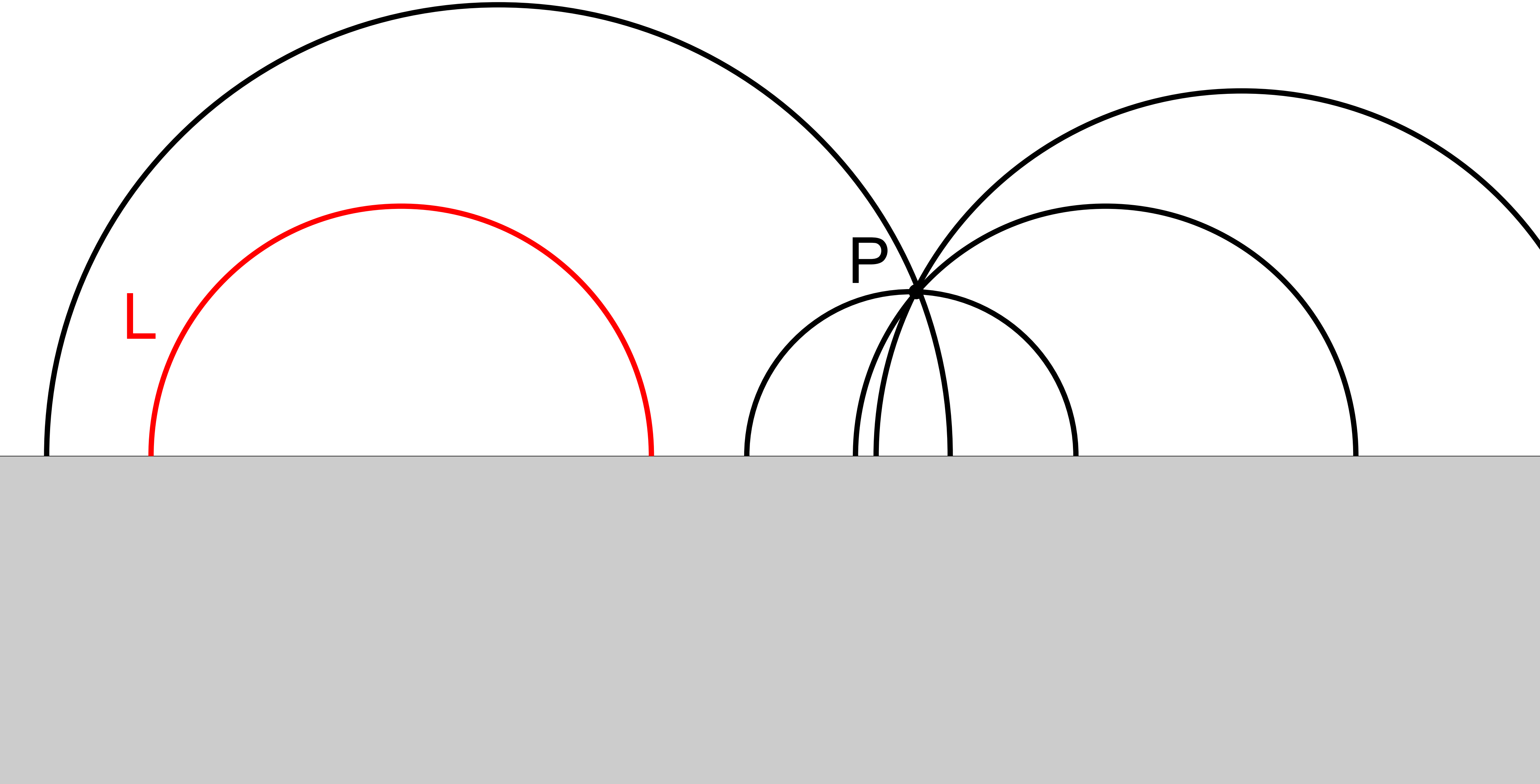
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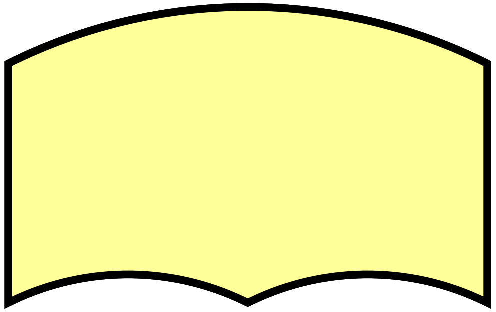
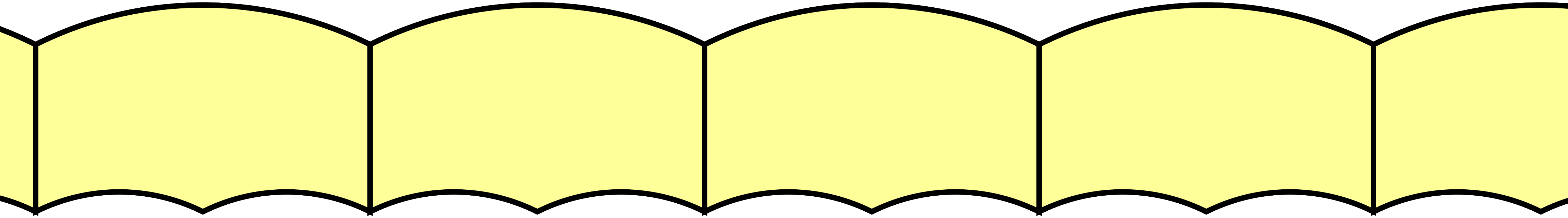
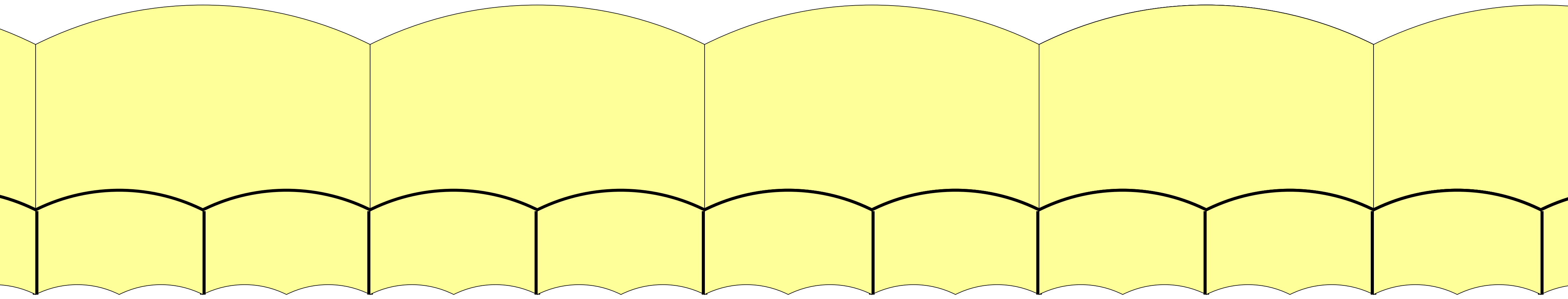


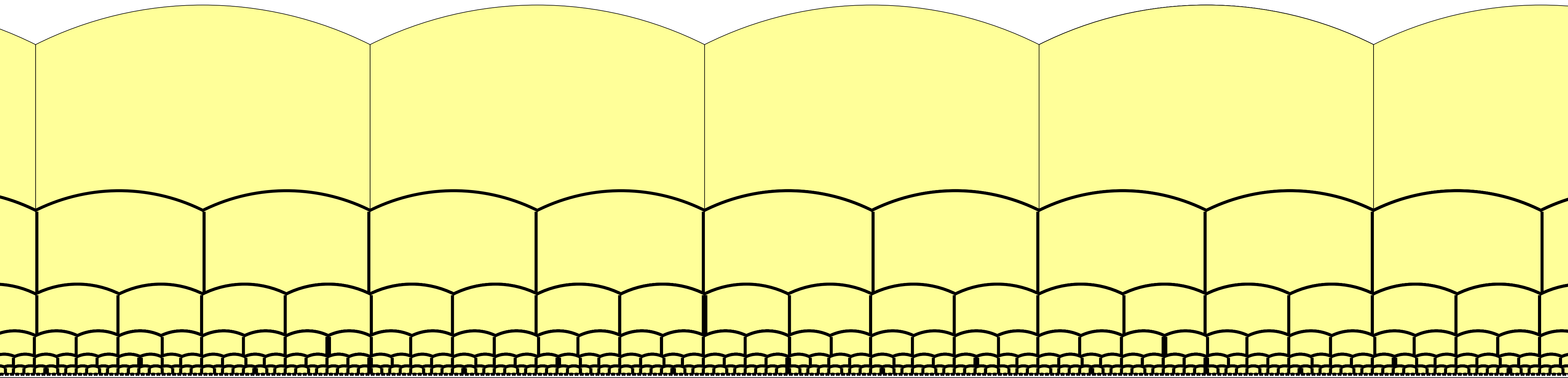
The role of the Euclidean Wang square tile will be played by a hyperbolic pentagon.



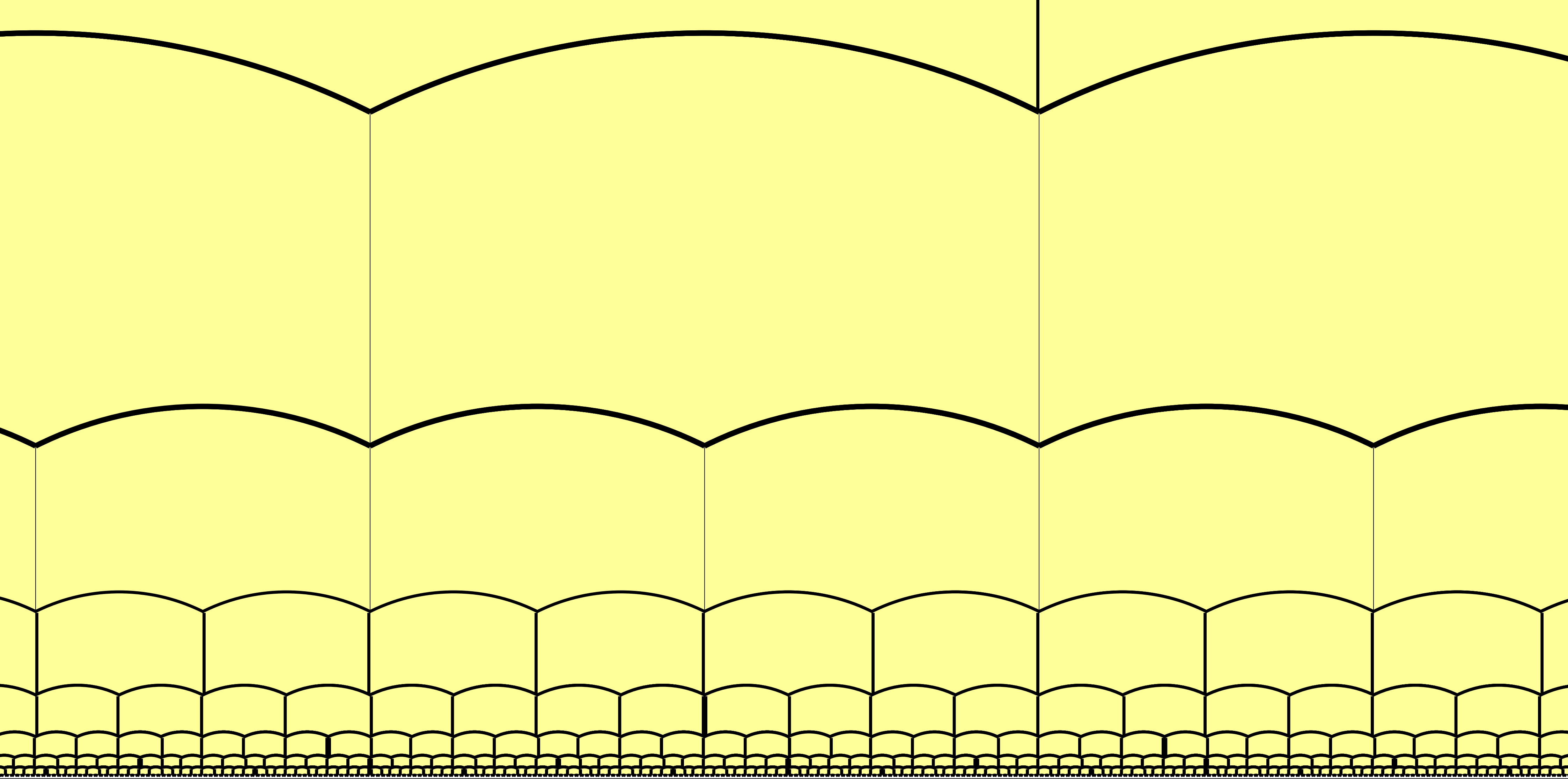
The pentagons can tile a "horizontal row".



”Beneath” each pentagon fits two identical pentagons. The pentagons are all congruent (=isometric copies of each other), but the projection makes objects close to the horizon seem smaller.

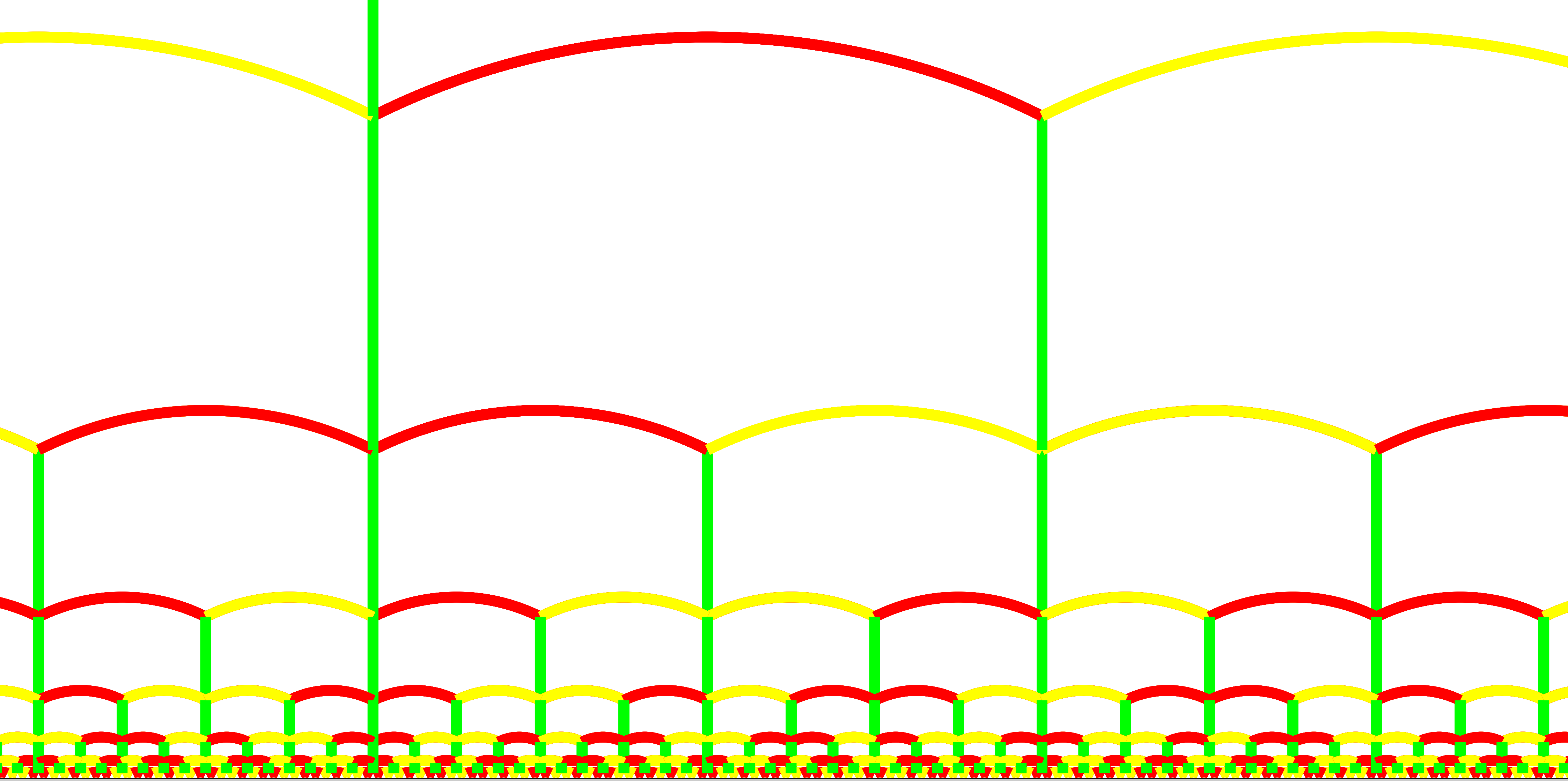


Infinitely many ”horizontal rows” fill the lower part of the half plane.



Similarily the upper part can be filled. We see that the pentagons tile the hyperbolic plane (in an uncountable number of different ways, in fact.)

On the hyperbolic plane Wang tiles are pentagons with colored edges. Such pentagons may be placed adjacent if the edge colors match. A given set of pentagons tiles the hyperbolic plane if a tiling exists where the color constraint is everywhere satisfied.



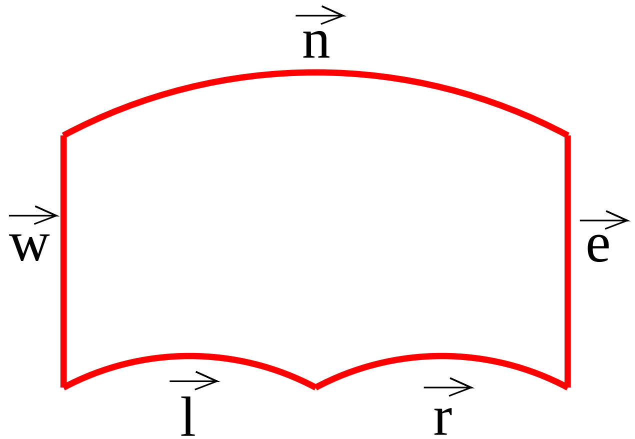
The two sample tiles admit a tiling.

The hyperbolic tiling problem asks whether a given finite collection of colored pentagons admits a valid tiling.

Theorem. The tiling problem of the hyperbolic plane is undecidable.

Note that the hyperbolic Wang tiles can be transformed into equivalent shapes exactly as in the Euclidean case: by introducing different bumps and dents for different colors. So the undecidability holds for the tiling problem using hyperbolic polygons.

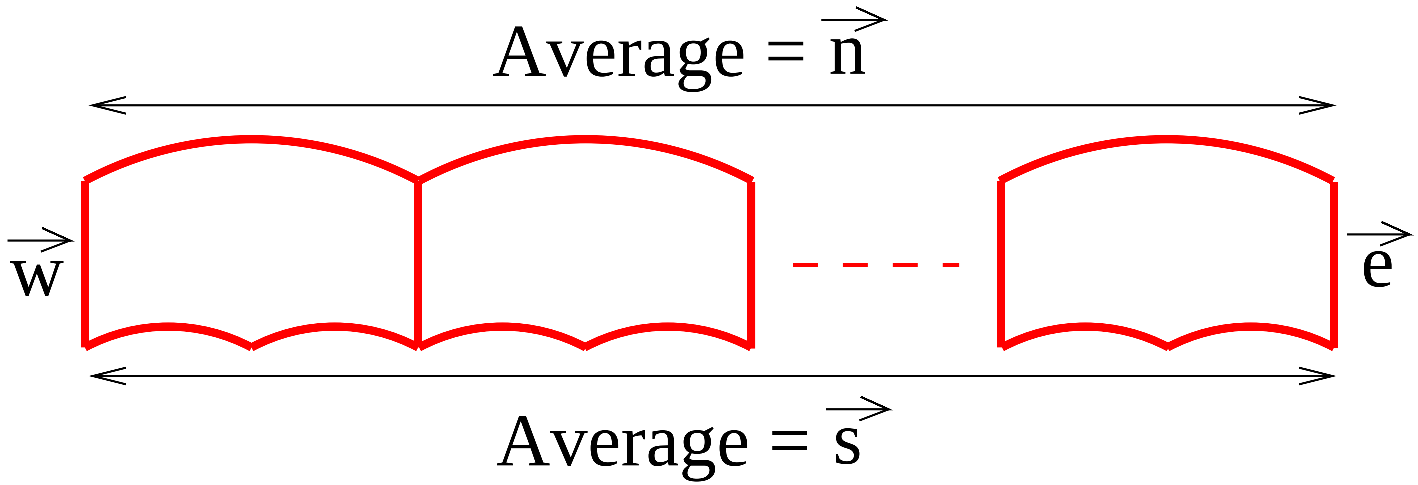
We say that pentagon



computes the affine transformation $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ if

$$f(\vec{n}) + \vec{w} = \frac{\vec{l} + \vec{r}}{2} + \vec{e}.$$

(Difference to Euclidean Wang tiles: The "output" is now divided between $\vec{l}$ and $\vec{r}$.)

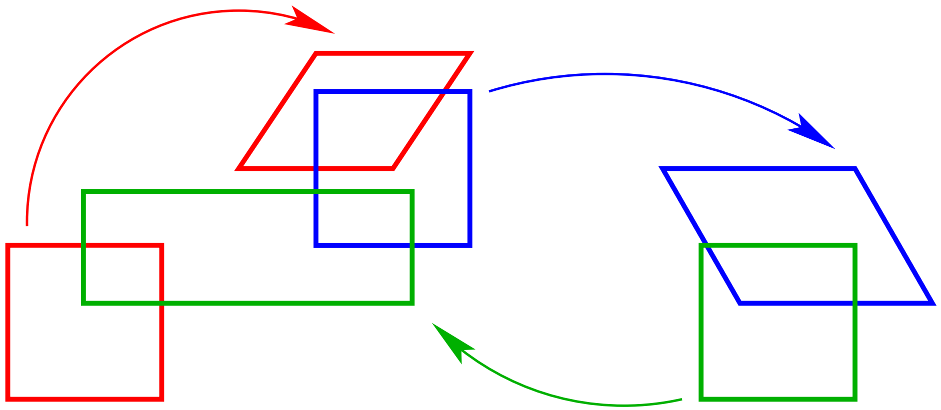


In a horizontal segment of length n where all tiles compute the same f holds

$$f(\vec{n}) + \frac{1}{n}\vec{w} = \vec{s} + \frac{1}{n}\vec{e},$$

where $\vec{n}$ and $\vec{s}$ are the averages of the top and the bottom labels.

As the segment is made longer, the effect of the carry in and out labels $\vec{w}$ and $\vec{e}$ vanish.



For a given system of affine maps f_i and unit squares U_i we construct for each i a set T_i of pentagons

- that compute function f_i, and
- whose top edge labels $\vec{n}$ are in U_i.

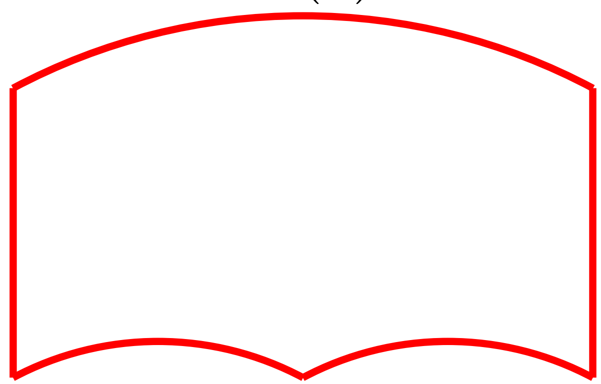
It follows, exactly as in the Euclidean case, that if a valid tiling of the hyperbolic plane with such pentagons exists then from the labels of horizontal rows one obtains an infinite orbit in the system of affine maps.

We still have to detail how to choose the tiles so that the converse is also true: if an immortal point exists then its orbit provides a valid tiling.

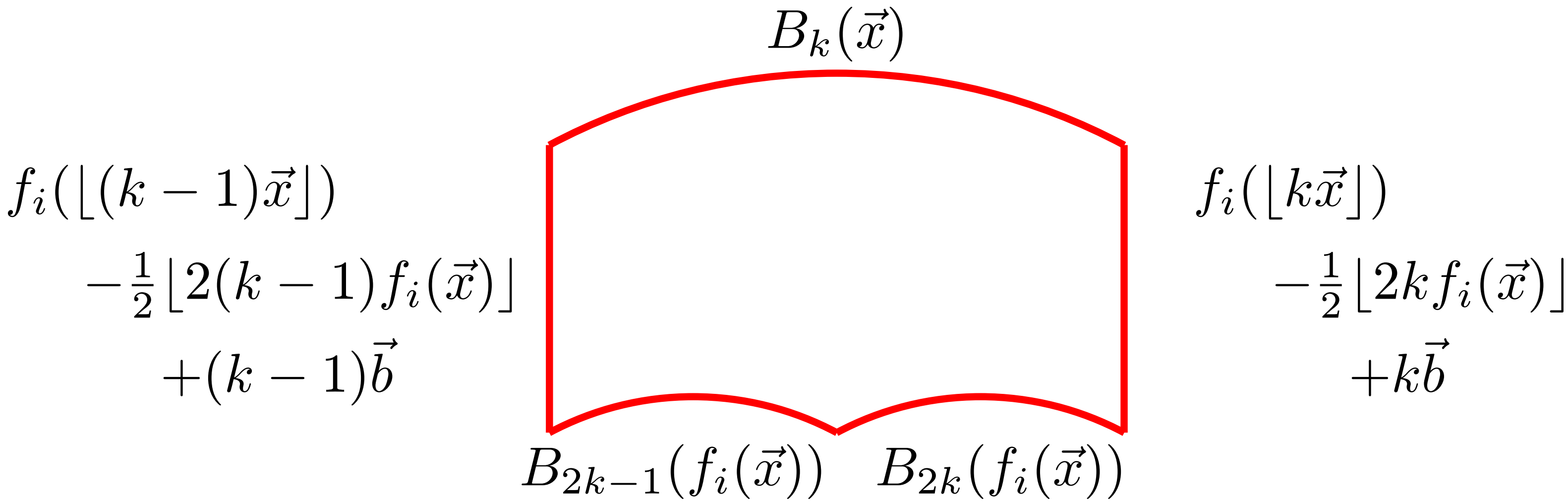
The tile set corresponding to a rational affine map

$$f_i(\vec{x}) = M\vec{x} + \vec{b}$$

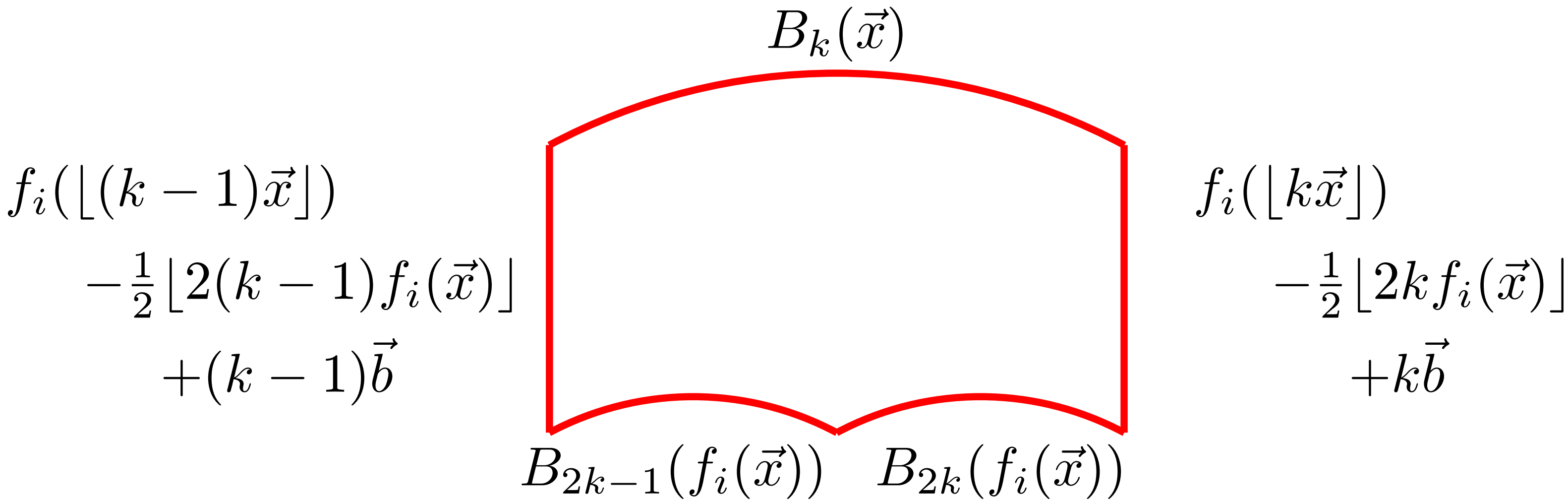
and its domain square U_i consists of all tiles



where $k \in \mathbb{Z}$ and $\vec{x} \in U_i$.

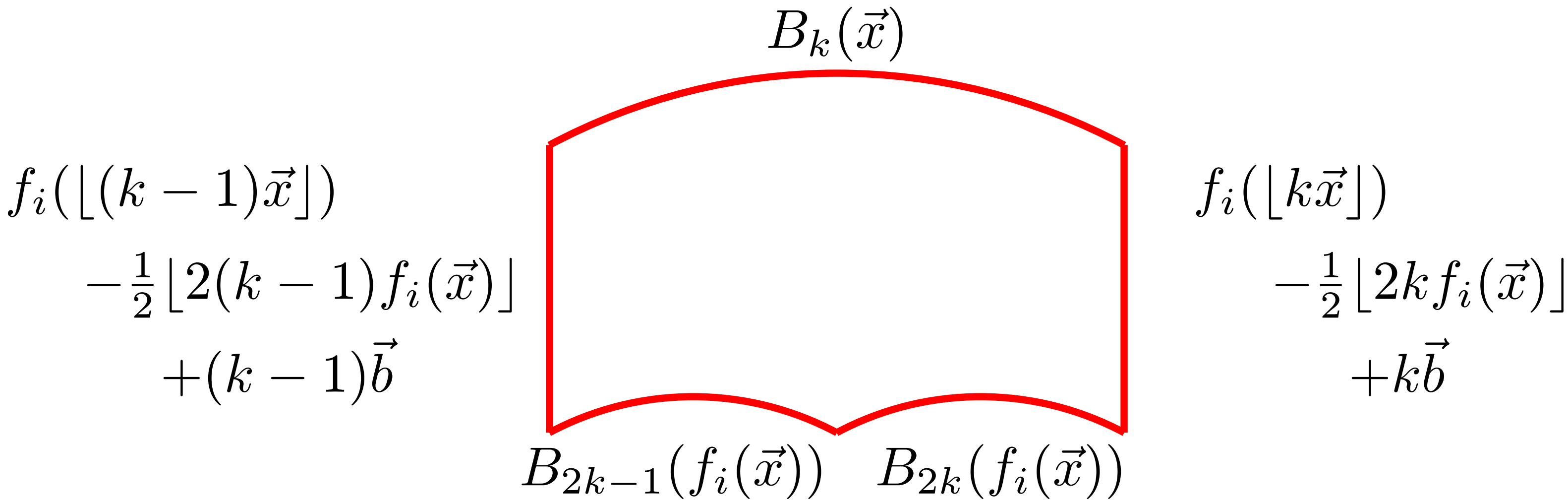


(1) For fixed $\vec{x} \in U_i$ the tiles for consecutive $k \in \mathbb{Z}$ match so that a horizontal row can be formed whose top and bottom labels read the balanced representations of $\vec{x}$ and $f_i(\vec{x})$, respectively.



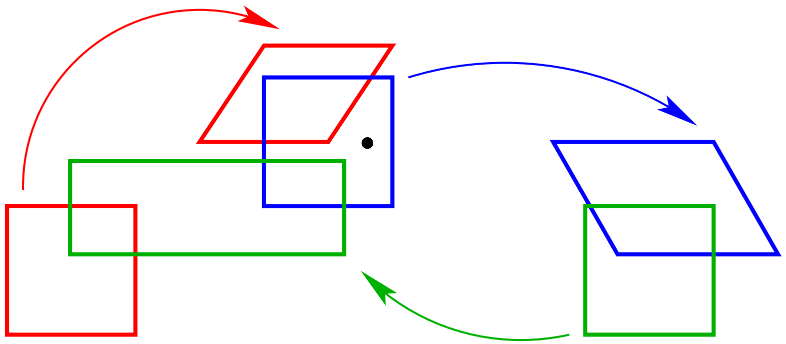
(2) A direct calculation shows that the tile computes function f_i:

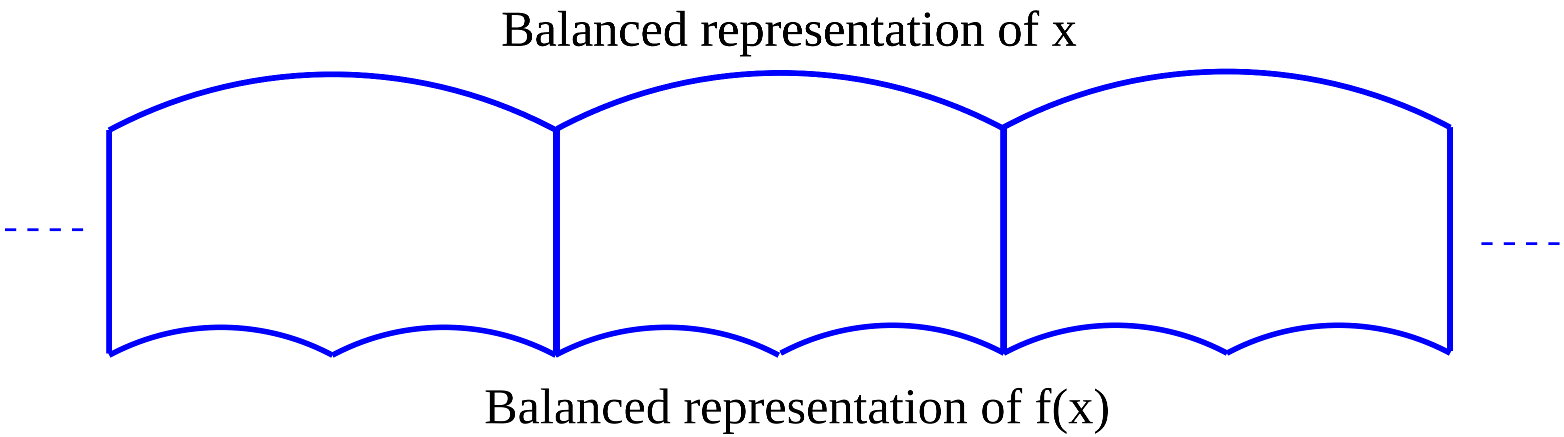
$$f_i(\vec{n}) + \vec{w} = \frac{\vec{l} + \vec{r}}{2} + \vec{e}.$$



(3) There are only finitely many such tiles (when f_i is rational), and they can be effectively constructed.

The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





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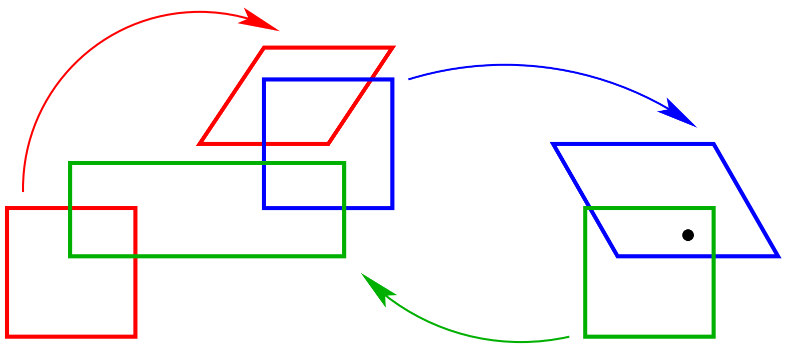
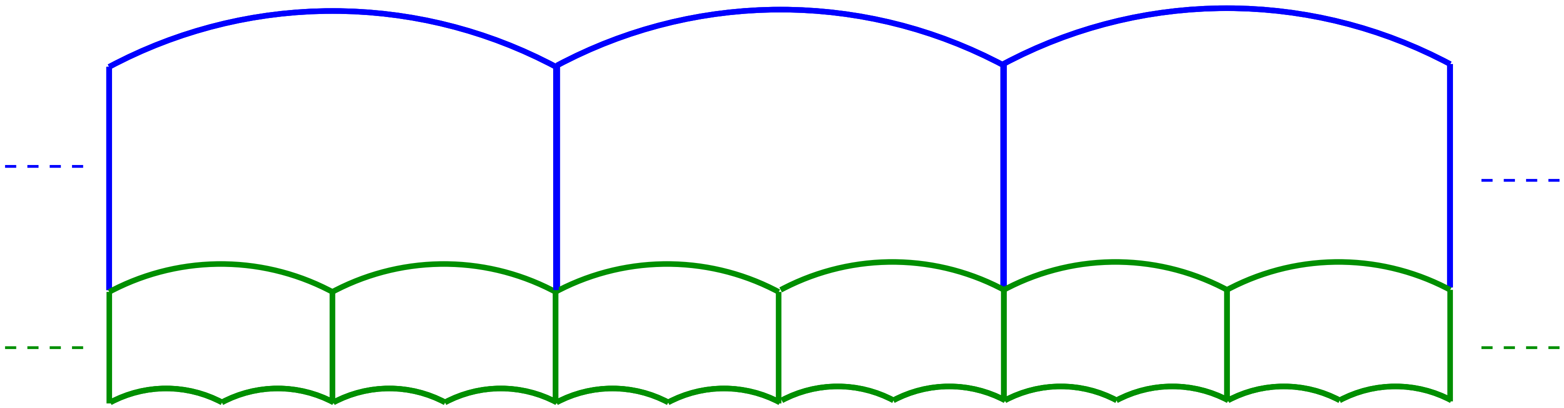
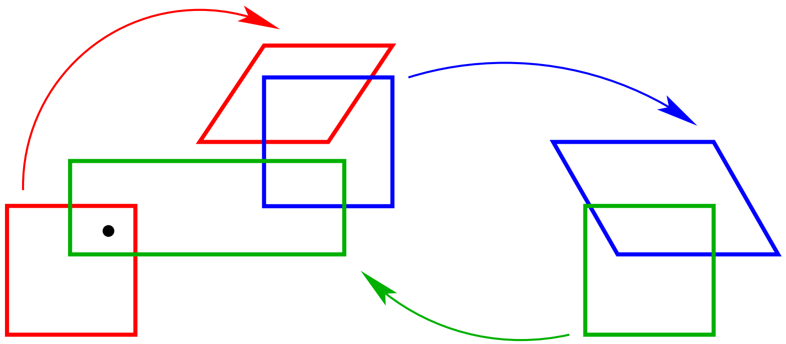


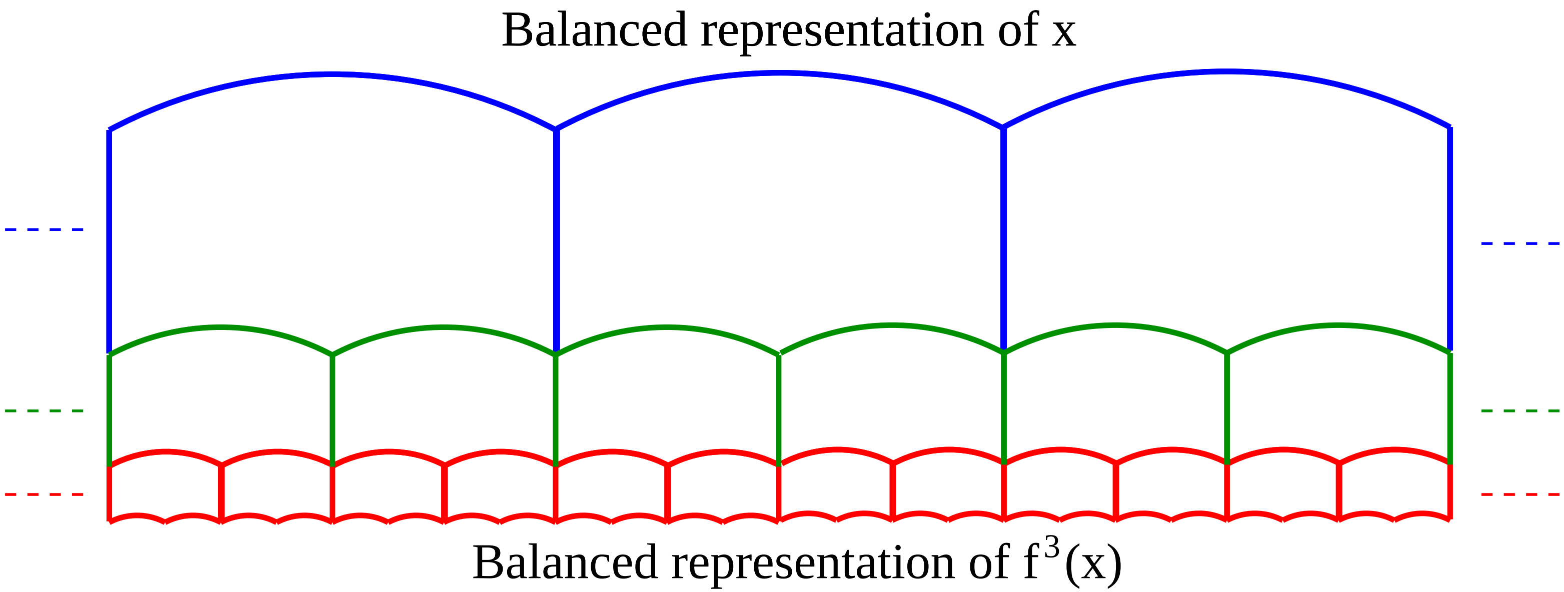
Balanced representation of x



Balanced representation of $f^2(x)$

The tiles constructed admit a valid tiling iff the system of affine maps has an immortal point:





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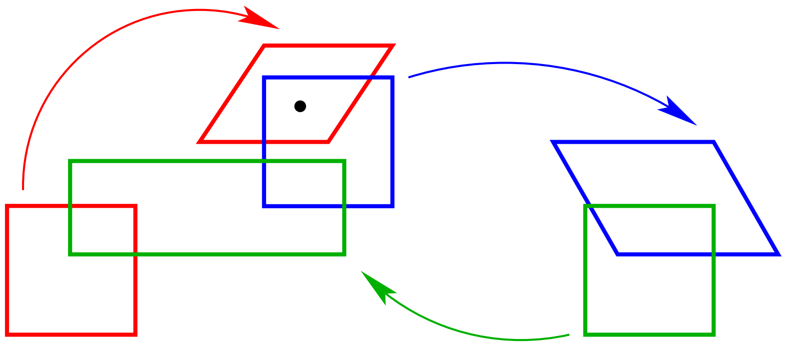
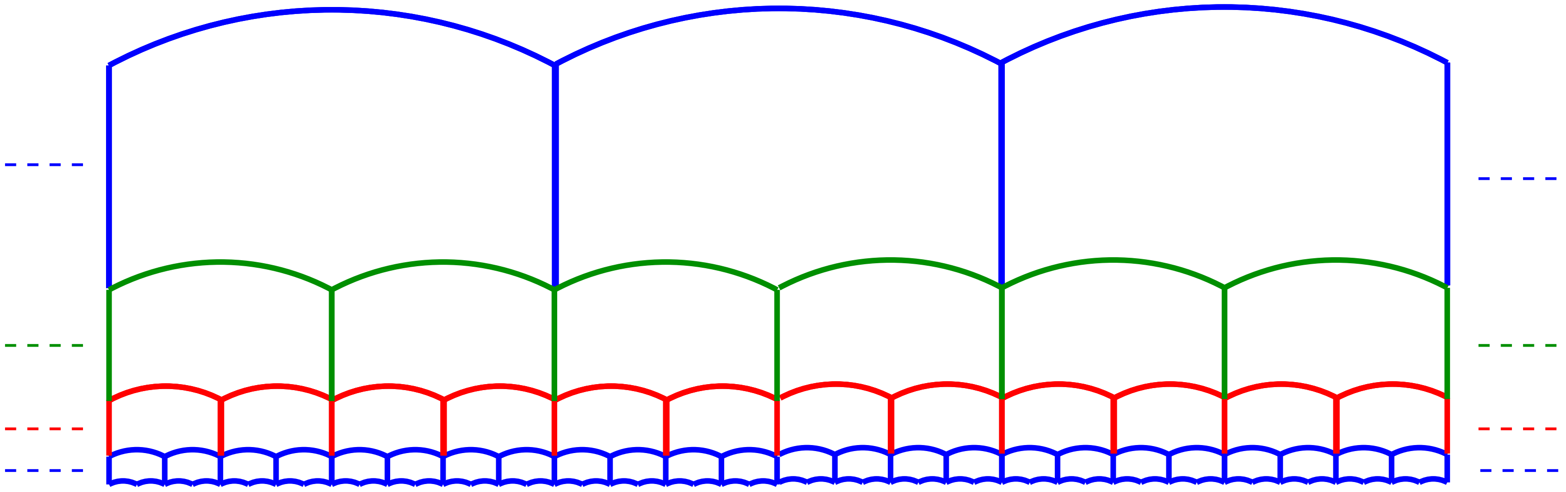


Balanced representation of x



Balanced representation of $f^4(x)$

Conclusion

A new proof for the undecidability of the tiling problem was presented. The proof was based on a reduction where one constructed tiles such that valid tilings are forced to simulate iterations of a system of affine transformations, which in turn simulate Turing machine computations.

The construction works well also in other set-ups. In particular, we showed that the tiling problem in the hyperbolic plane is undecidable.

One can also obtain aperiodic tile sets on the hyperbolic plane. (Aperiodicity: no tiling is left invariant by any non-trivial isometry of the plane.) A set of 15 hyperbolic tiles comes up easily from the construction.

Future work

What other lattices (e.g. Cayley graphs of which finitely generated groups) have undecidable tiling problem ?

Can the Turing machine simulations presented in this work be done in one-dimensional cellular automata ? In other words, it would be nice to have a CA simulate a Turing machine uniformly everywhere in space, so that any segment of a CA configuration encodes a segment of the Turing machine tape, and longer CA segments encode longer portions of the Turing machine tape.

What other tiling properties can be deduced undecidable using this method ?

What is the smallest aperiodic set of Wang tiles ?